QUANTUM MECHANICS
Lecture 26
Addition of angular momenta

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D. J. Griffiths: paragraph 4.4.3

In Classical Mechanics . . .

1. In Classical Mechanics, if $\vec{L}_1$ and $\vec{L}_2$ are the angular momenta of two parts of a system, its total angular momentum

$$\vec{L} = \vec{L}_1 + \vec{L}_2$$

has a modulus square given by

$$|\vec{L}|^2 \equiv L^2 = L_1^2 + L_2^2 + 2\, L_1\, L_2 \, cos\theta$$

and therefore

$$|L_1 - L_2| \leq L \leq L_1 + L_2$$

2. What happens in Quantum Mechanics ?

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2. **What happens in Quantum Mechanics ?**

In Quantum Mechanics . . .

1. To better focalize the problem, let us assume to deal with a system which has an angular momentum L and a spin S.
2. Which are the possible values of the total angular momentum J of the system ?
3. By hypothesis, the Hilbert space $\mathcal{H}$ admits an orthonormal basis made by the vectors $|l, m; s, s_z>$, simultaneous eigenvectors of the operators L^2, L_z, S^2, S_z for the eigenvalues

$$L^2 \Rightarrow l(l+1)\,\hbar^2; \qquad L_z \Rightarrow m\,\hbar$$
$$S^2 \Rightarrow s(s+1)\,\hbar^2; \qquad s_z \Rightarrow m\,\hbar$$

and the dimension of $\mathcal{H}$ is

$$N = (2l+1)(2s+1)$$

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In Quantum Mechanics ...

1. The total angular momentum operator J is defined as

$$J \equiv (J_x, J_y, J_z) = (L_x + s_x, L_y + s_y, L_z + s_z)$$

   and it operates inside the space $\mathcal{H}$.

2. Since the commutation relations concerning J are the same of L and S, we can already conclude that the only possible eigenvalues of J^2 will have the form $j(j+1)\hbar^2$ with j integer or half-integer.
   Moreover, for a given j, there must be $2j+1$ states with that j and $j_z = M\hbar$, where $-j \leq M \leq j$.

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In Quantum Mechanics ...

The vectors $|l, m; s, s_z>$ are eigenvectors of J_z, in fact

$$\begin{aligned} J_z|l, m; s, s_z> &\equiv (L_z + S_z)|l, m; s, s_z>= \\ &= (m + s_z)\hbar|l, m; s, s_z> \end{aligned}$$

but, in general, they **are not** eigenvectors of J^2 because

$$J^2 = L^2 + S^2 + 2\vec{L} \cdot \vec{S}$$

and the states $|l, m; s, s_z>$ are eigenvectors of L^2 and S^2, but **not** of

$$2\vec{L} \cdot \vec{S} = 2(L_x S_x + L_y S_y + L_z S_z)$$

but only of $L_z S_z$ ...

In Quantum Mechanics . . .

- Since the $|l, m; s, s_z>$ are a basis and eigenvectors of J_z for the eigenvalue $(m + s_z)\hbar$, clearly the highest possible eigenvalue of J_z will be $j_z = \hbar(l + m)$, corresponding to $|l, l; s, s>$.
- This means that one of the possible values of j must be $j = l + s$ and **no higher** j can be possible because it would imply a possible j_z higher that $\hbar(l + m)$.
- In this way, we have already determined the highest possible j and the **unique** state $|l, l; s, s> \equiv |j, j>$ corresponding to $J_z = j\,\hbar$.

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- If we make use, now, of the lowering operator $J_- \equiv L_- + S_-$, we can define the full chain made by $2j + 1$ independent vectors, characterized by the value of $j = l + s$ and J_z with eigenvalue $M\hbar$ such that $-j \leq M \leq j$.
- However, if for $M = l + s$ we have only one possible eigenstate in $\mathcal{H}$, for $M = l + s - 1$ we have **two** independent eigenstates

$$|l, l-1; s, s>; \quad |l, l; s, s-1>$$

and for $M = l + s - 2$ the independent eigenstates in $\mathcal{H}$ are **three**

$$|l, l-2; s, s>; \; |l, l-1; s, s-1>; \; |l, l; s, s-2>$$

and their number increases up to $M = |l - s|$.

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- Let us point out, however, that the only state with $M = l + s - 1$ belonging to the chain defined by $|j, j> \equiv |l, l; s, s>$ is

$$|j, j-1> = J_-|j, j> = (L_- + S_-)|l, l; s, s> =$$
$$= \sqrt{2l}|l, l-1; s, s> + \sqrt{2s}|l, l; s, s-1>$$

- The other J_z eigenvector with $M = l + s - 1$, orthogonal to $|j, j-1>$, is indeed the top head of a new chain of vectors characterized by $j = l + s - 1$ and eigenvalue of J_z equal to $M\hbar = (l + s - 1)\hbar$.
This second chain is characterized by a J^2 eigenvalue of $[(j-1)j]\hbar^2$ and eigenvalues of J_z spanning from $-(l + s - 1)\hbar$ up to $(l + s - 1)\hbar$, in step of $\hbar$.

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- But we have seen that, for $M = l + s - 2$, the independent eigenvectors of J_z are indeed three. We have also seen that one belongs to the chain characterized by $j = l + s$ and another to the chain characterised by $j = l + s - 1$.
- The third one is a top head of a chain of eigenvectors of J^2 and J_z with $j = l + s - 2$.
- The process continues until we arrive to $j = |l - s|$ where it stops.

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Angular momenta composition

Example:
composition of $L = 2$ with $S = 1$ (in $\hbar$ units).
The total number of states is
$(2L + 1) \cdot (2S + 1) = 5 \cdot 3 = 15$.
The possible values of the total angular momentum J run from $|L - S| = 1$ up to $L + S = 3$.

M	l,s		
-3	-2, -1		
-2	-2, 0	-1, -1	
-1	-2, 1	-1, 0	0, -1
0	0, 0	-1, 1	1, -1
1	2, -1	1, 0	0, 1
2	2, 0	1, 1	
3	2, 1		

We have
J=3: 7 states
J=2: 5 states
J=1: 3 states

Angular momenta composition

The vectors $|J, M>$ in the basis $|L, m; S, s_z>$ ($\equiv$ for short $(m; s_z)$) are as follows:

$$
\begin{aligned}
|3,3> &= (2;1) \\
|3,2> &= \sqrt{\frac{1}{3}}(2;0) + \sqrt{\frac{2}{3}}(1;1) \\
|3,1> &= \sqrt{\frac{1}{15}}(2;-1) + \sqrt{\frac{8}{15}}(1;0) + \sqrt{\frac{6}{15}}(0;1) \\
|3,0> &= \sqrt{\frac{1}{5}}(1;-1) + \sqrt{\frac{3}{5}}(0;0) + \sqrt{\frac{1}{5}}(-1;1) \\
|3,-1> &= \sqrt{\frac{1}{15}}(-2;1) + \sqrt{\frac{8}{15}}(-1;0) + \sqrt{\frac{6}{15}}(0;-1) \\
|3,-2> &= \sqrt{\frac{1}{3}}(-2;0) + \sqrt{\frac{2}{3}}(-1;-1)
\end{aligned}
$$

Angular momenta composition

$J = 2$

$$|2,2> \;=\; \sqrt{\frac{2}{3}}(2;0) - \sqrt{\frac{1}{3}}(1;1)$$

$$|2,1> \;=\; \sqrt{\frac{2}{6}}(2;-1) + \sqrt{\frac{1}{6}}(1;0) - \sqrt{\frac{3}{6}}(0;1)$$

$$|2,0> \;=\; \sqrt{\frac{1}{2}}(1;-1) - \sqrt{\frac{1}{2}}(-1;1)$$

$$|2,-1> \;=\; -\sqrt{\frac{2}{6}}(-2;1) - \sqrt{\frac{1}{6}}(-1;0) + \sqrt{\frac{3}{6}}(0;-1)$$

$$|2,-2> \;=\; -\sqrt{\frac{2}{3}}(-2;0) + \sqrt{\frac{1}{3}}(-1;-1)$$

Angular momenta composition

$J = 1$

$$
\begin{aligned}
|1,1> &= \sqrt{\frac{6}{10}}(2;-1) - \sqrt{\frac{3}{10}}(1;0) + \sqrt{\frac{1}{10}}(0;1) \\
|1,0> &= \sqrt{\frac{3}{10}}(1;-1) - \sqrt{\frac{4}{10}}(0;0) + \sqrt{\frac{3}{10}}(-1;1) \\
|2,-1> &= \sqrt{\frac{6}{10}}(-2;1) - \sqrt{\frac{3}{10}}(-1;0) + \sqrt{\frac{1}{10}}(0;-1)
\end{aligned}
$$

The coefficients that allow to express the vectors $|J, M>$ in the basis $|L, m; S, S_z>$ are the so-called "Clebsch-Gordan" coefficients.

Angular momenta composition

- Another interesting case to be considered is the composition of two spins $S = \frac{1}{2}$.
- In this case, the Hilbert space is made by $(2S+1)(2S+1) = 4$ states and, according to the rule for which
  $|S_1 - S_2| \leq J \leq S_1 + S_2$
  the only possible values of the total angular momentum are $J = 0$ (singlet) and $J = 1$ (triplet).

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Angular momenta composition

In terms of $|\frac{1}{2}, S_{1z}; \frac{1}{2}, S_{2z} > \equiv (S_{1z}; S_{2z})$, the eigenvectors $|j, m>$ are

$$|1,1> \; = \; (+;+)$$
$$|1,0> \; = \; \sqrt{\frac{1}{2}}\Big[(+;-) + (-;+)\Big]$$
$$|1,-1> \; = \; (-;-)$$

$$|0,0> \; = \; \sqrt{\frac{1}{2}}\Big[(+;-) - (-;+)\Big]$$

where $\pm$ stands for $\pm\frac{1}{2}(\hbar)$.
Under the exchange of the two spins, the states of the triplet ($J = 1$) are symmetric, whereas the state of singlet ($J = 0$) is antisymmetric.