QUANTUM MECHANICS
Lecture 25

The Spin
The spin 1/2

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D. J. Griffiths: paragraph 4.4

The spin

In Classical Mechanics, the **total angular momentum** of a rigid body is the **sum of two terms:**

- the ***orbital momentum*** $\vec{L} = \vec{r} \times \vec{p}$, associated with the center of mass linear momentum $\vec{p}$ and the center of mass position $\vec{r}$, in the inertial reference system that we have choosen;
- the ***spin***, which is the angular momentum of the different parts of the rigid body with respect to its center of mass $\vec{S} = I\,\vec{\omega}$.

The spin

1. In Quantum Mechanics we have a **similar situation**, but with a **fundamental difference** that we will point out in a moment.
2. Let us take, for instance, the case of the hydrogen atom.
3. The electron **total angular momentum** is in fact the **sum** of its **orbital momentum** $\vec{L}$, associated to its motion around the nucleus (assumed at rest) and described by the spherical harmonics, with an ***intrinsic angular momentum***, **its spin** $\vec{S}$, which **has nothing to do** with any effective spinning of the particle which, as far as we know, **is a structureless point-like particle !**

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The spin

As a matter of fact, the spin $\vec{S}$ of an elementary particle, like the electron, is a **completely real novelty**, present in Quantum Mechanics but **without any classical analogue.**

The spin

1. By definition, the **commutation relations** of the spin operator components S_x, S_y, S_z are the **same** as those concerning the **orbital momentum** components L_x, L_y, L_z:

$$[S_x, S_y] = i\hbar S_z; \quad [S_y, S_z] = i\hbar S_x; \quad [S_z, S_x] = i\hbar S_y$$

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The spin

1. If we use the ket Dirac notation $|s,m>$ to describe the simultaneous eigenvectors of S^2 and S_z, by hypothesis we have

$$\begin{aligned} S^2|s,m> &= \hbar^2\, s(s+1)\,|s,m> \\ S_z|s,m> &= \hbar\, m\,|s,m> \end{aligned}$$

with $s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \ldots$ and $m = -s,\ -s+1,\ \ldots .s-1,\ s.$

2. Moreover, it can be shown that the operators $S_\pm \equiv S_x \pm iS_y$ are such that

$$S_\pm|s,m> = \hbar\sqrt{s(s+1)-m(m\pm 1)}\ \ |s,m\pm 1>$$

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1. In this case, however, the eigenvectors $|s, m>$ cannot be represented with the spherical harmonics because **they have** **nothing to do with the angular variables.**
2. As a matter of fact, the vectors $|s, m>$ are necessary to complement the description of the physical state in terms of wave functions $\psi(\vec{r})$, without any connection to the position variables $\vec{r}$ (or r, θ, ϕ).

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Spin 1/2

1. The most important case concerning the spin is $S = \frac{1}{2}$. This is, in fact, the spin of the electron, proton, neutron, quarks, etc... .
2. For this spin, there are only two eigenstates of S_z:
   $|s = \frac{1}{2}, m = \frac{1}{2} > \equiv |\frac{1}{2}, \frac{1}{2} >$ (spin up $\Uparrow$)
   $|s = \frac{1}{2}, m = -\frac{1}{2} > \equiv |\frac{1}{2}, -\frac{1}{2} >$ (spin down $\Downarrow$).
3. The general spin state is a linear combination of the two eigenstates which, therefore, can be represented by a two-element column matrix (spinor)

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv$$
$$\equiv \alpha|1/2, 1/2 > + \beta|1/2, -1/2 >$$

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1. The spin Hilbert space is two-dimensional and the spin operators are represented by 2×2 matrices.
2. The S^2 operator has the only eigenvalue $\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2$, therefore it is multiple of the identity matrix:

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whereas, in the basis already mentioned of the simultaneous eigenvectors of S^2 and S_z, we have

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Spin 1/2

Concerning the "ladder" operators $S_\pm = S_x \pm iS_y$, from their definition we have

$$\begin{aligned} S_+|1/2,1/2> &= 0; \\ S_+|1/2,-1/2> &= \hbar|1/2,1/2> \end{aligned}$$

$$\begin{aligned} S_-|1/2,1/2> &= \hbar|1/2,-1/2>; \\ S_-|1/2,-1/2> &= 0 \end{aligned}$$

and therefore

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Spin 1/2

from which we have

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{-i}{2}(S_+ - S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Spin 1/2

Usually, the operators S_x, S_y, S_z are written in terms of the **Pauli matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

Spin 1/2

The most relevant properties of the Pauli matrices $(\sigma_x,\ \sigma_y,\ \sigma_z) \equiv (\sigma_1,\ \sigma_2,\ \sigma_3)$ are the following

- $\sigma_i = (\sigma_i)^\dagger$;
- $(\sigma_i)^2 = I$;
- $\sigma_i\,\sigma_j = -\sigma_j\,\sigma_i$ when $i \neq j$;

and

$$
\begin{aligned}
[\sigma_x, \sigma_y] &\equiv [\sigma_1, \sigma_2] = 2i\,\sigma_3 \equiv 2i\,\sigma_z \\
[\sigma_y, \sigma_z] &\equiv [\sigma_2, \sigma_3] = 2i\,\sigma_1 \equiv 2i\,\sigma_x \\
[\sigma_z, \sigma_y] &\equiv [\sigma_3, \sigma_2] = 2i\,\sigma_2 \equiv 2i\,\sigma_y
\end{aligned}
$$

Spin 1/2

Concerning the description of the physical state of a particle with spin $s = 1/2$, the wave-function $\Psi(x)$, that we have considered so far, **is not anymore sufficient**, because it has to do only with the spatial variables that have nothing to do with the spin.

We need now a two-component wave function

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$$

where the components $\psi_\pm$ specify the spatial content of the $S_z = \pm\hbar/2$ spin states.
Concerning the normalization, we must have

$$1 = \int dx\, \Psi^\dagger(x)\Psi(x) \equiv \int dx \Big(|\psi_+(x)|^2 + |\psi_-(x)|^2 \Big)$$