QUANTUM MECHANICS Lecture 24

Still about the angular momentum

Enrico Iacopini

December 3, 2019

D. J. Griffiths: paragraph 4.3

The angular momentum

1. In the previous lecture we have seen that the operators L^2 and L_z commute, which means that they are **compatible observables**.

2. We can, therefore, **look for simultaneous eigenfunctions of the two operators:**

$$L^2 f = \lambda\, f \quad and \quad L_z f = \mu\, f$$

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To solve the eigenvalue equations for L^2 and L_z, let us start by defining the following two (non hermitian) operators $L_\pm \equiv L_x \pm i\, L_y$.

We have

$$
\begin{aligned}
[L_z, L_\pm] &= [L_z, L_x] \pm i\,[L_z, L_y] = \\
&= i\hbar\, L_y \pm i(-i\hbar\, L_x) = i\hbar\, L_y \pm \hbar\, L_x = \\
&= \pm\hbar(L_x \pm i\, L_y) = \pm\hbar\, L_\pm
\end{aligned}
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The angular momentum

1. Now, since, $[L^2, L_\pm] = [L^2, L_x \pm iL_y] = 0$, if f is an eigenfunction of L^2 for the eigenvalue λ, also $L_\pm f$ **has the same property.** In fact

$$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f)$$

2. Concerning L_z, we have instead that

$$\begin{aligned} L_z(L_\pm f) &= (L_z L_\pm - L_\pm L_z)f + L_\pm(L_z f) = \\ &= (\pm\hbar L_\pm)f + L_\pm(\mu\, f) = \\ &= (\mu \pm \hbar)L_\pm f \end{aligned}$$

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The angular momentum

1. We call L_+ and L_- ***raising*** and ***lowering*** (*ladder*) operators, respectively.

2. According to the previous conclusions, starting from the eigenfunction f, corresponding to the eigenvalue λ of L^2 and μ of L_z, with the raising operator f.i. we can build the functions L_+f, $(L_+)^2f$, ... which are eigenfunctions of L_z for the eigenvalues $\mu + \hbar$, $\mu + 2\hbar$, ..., remaining eigenfunctions of L^2 for the initial eigenvalue λ.

3. **This "raising" chain will stop somewhere or not ?**

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1. As a matter of fact, since the L_z eigenfunction $(L_+)^n f$ is eigenfunction of L^2 for the eigenvalue λ, the **chain cannot continue indefinitely** because, on any function (and therefore also on $(L_+)^n f$ ...) we must have

$$< L^2 > \;\geq\; < L_z^2 > \;\Rightarrow\; \lambda \geq (\mu + n\hbar)^2$$

so, to stop the chain, there must be a "top" L_z eigenvector f_t for which $L_+ f_t = 0$.

2. Let $\hbar\, l_t$ be the highest eigenvalue of L_z for the given eigenvalue λ of L^2. This means that there exists a function $f_t \neq 0$ such that

$$L_z\, f_t = \hbar l_t\; f_t; \quad L^2 f_t = \lambda\, f_t; \quad L_+ f_t = 0$$

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\begin{aligned}
L_\pm L_\mp &= (L_x \pm i L_y)(L_x \mp i L_y) = \\
&= L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) = \\
&= L_x^2 + L_y^2 \mp i(i\hbar L_z) = L^2 - L_z^2 \pm \hbar\, L_z \\
&\Rightarrow L^2 = L_\pm L_\mp + L_z^2 \mp \hbar\, L_z
\end{aligned}
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2. therefore, using this result on f_t, we have

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\begin{aligned}
L^2 f_t &= (L_- L_+ + L_z^2 + \hbar\, L_z) f_t \\
&\Rightarrow \lambda = (\hbar\, l_t)^2 + \hbar(\hbar\, l_t) = \hbar^2 (l_t^2 + l_t) \\
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1. For the same reason for which the raising chain must stop somewhere, also the lowering chain produced by the operators $(L_-)^n$ must do the same, because, again, we have to satisfy the condition

$$< L^2 > \ \geq \ < L_z^2 > \ \Rightarrow \ \lambda \geq (\mu - n\hbar)^2$$

2. Let $\hbar\, l_b$ be the lowest eingenvalue of L_z for the given eigenvalue λ of L^2. This means that there exists a function $f_b \neq 0$ such that

$$L_z f_b = \hbar l_b \ f_t; \quad L^2 f_b = \lambda \ f_b; \quad L_- f_b = 0$$

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$$\begin{aligned} L^2 f_b &= (L_+ L_- + L_z^2 - \hbar\, L_z) f_b \\ &\Rightarrow \lambda = (\hbar\, l_b)^2 - \hbar(\hbar\, l_b) = \hbar^2 (l_b^2 - l_b) \\ &\Rightarrow \lambda = \hbar^2\, l_b (l_b - 1) \end{aligned}$$

2. The two equations that we have found concerning l_t, l_b and λ say that

$$\frac{\lambda}{\hbar^2} = l_t(l_t + 1) = l_b(l_b - 1)$$

which implies that

$$l_t = -l_b \equiv l \geq 0 \quad \Rightarrow \quad \lambda = l(l+1)\,\hbar^2$$

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As far as the eigenvalues of L_z, clearly they are such that

$$\mu = m\hbar; \quad m = -l, -l+1, \dots l-1, l$$

so, for a given l, they are in total $N = 2l + 1$, where N is integer, which implies that l (and therefore also m ...) must be **integer or half-integer.**

The angular momentum

1. In conclusion, the **simultaneous eigenfunctions** of the observables L^2 and L_z are characterized by two quantum numbers l, m such that

$$L^2 f = \hbar^2 l(l+1) f; \quad L_z f = \hbar m f$$

with

$$\begin{aligned} l &= 0, \frac{1}{2}, 1, \ldots \\ m &= -l, -l+1, \ldots l-1, l \end{aligned}$$

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The eigenfunctions

1. To find the explicit form of the simultaneous eigenfunctions of L^2 and L_z we need to write these operators in spherical coordinates.

2. It can be shown that

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$L^2 = -\hbar^2 \left[\frac{1}{sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right] = -\hbar^2 \hat{J}$$

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The eigenfunctions

1. Therefore, the eigenvalue equation for L^2 reads

$$L^2 f_l^m = \hbar^2 \, l(l+1) \, f_l^m$$
$$\Rightarrow \hat{J} \, f_l^m = -l(l+1) \, f_l^m$$

2. **But we have already seen this equation !**

3. In fact, this equation was already found when we have operated the separation of variables in the 3D time-independent Schrödinger equation.

4. **Its solutions are the spherical harmonics** $Y_l^m(\theta, \phi)$.

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1. Let us recall the definition of the $Y_l^m(\theta, \phi)$:

$$Y_l^m(\theta, \phi) \equiv \epsilon \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}}\, P_l^m(cos\theta)\, e^{im\phi}$$

   where l and m are integers, m is such that $|m| \leq l$, $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = 1$ for $m < 0$.

2. Clearly, these functions are also eigenfunctions of $L_z = -i\hbar\frac{\partial}{\partial\phi}$, in fact

$$-i\hbar\frac{\partial}{\partial\phi} Y_l^m(\theta, \phi) = m\hbar\, Y_l^m(\theta, \phi)$$

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The eigenfunctions

We can, therefore, conclude that, when we have solved the time-independent Schrödinger equation in $3D$ by separation of variables in radial and polar coordinates, we were indeed constructing **simultaneous eigenfunctions of the three commuting operators** H, L^2 and L_z

$$H\,\psi = E\,\psi;\;\; L^2\,\psi = \hbar^2\,l(l+1)\psi;\;\; L_z\,\psi = \hbar m\,\psi$$

The eigenfunctions

A final **observation** before leaving the subject:

the simultaneous eigenfunctions of L^2 and L_z, which we have seen to be the spherical harmonics, admit only values of l (and m) which are **integers**, whereas the algebraic theory previously developed, allows, in principle, also **half-integers** . . .

Which is the meaning af the half-integer solutions ?