QUANTUM MECHANICS
Lecture 20

Compatible and incompatible observables
The uncertainty principle revisited

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D. J. Griffiths: paragraphs 3.4, 3.5

Continuous spectrum

Together with the momentum $\hat{p}$, another important operator which has a **continuous spectrum is the position operator** $\hat{x}$.

Continuous spectrum

1. Its eigenfunction $g_y(x)$, corresponding to the eigenvalue $y \in R$, by definition, is such that

$$\hat{x}\, g_y(x) \equiv x \cdot g_y(x) = y\, g_y(x)$$

where x is the position variable, whereas y is the given eigenvalue of the operator $\hat{x}$.

2. The only possibility to satisfy the above eigenvalue equation is that the function $g_y(x)$ **is always zero except in** x**.**

To satisfy the generalized condition concerning its **finite** (*and not always null*) scalar product with any w.f., we must have

$$g_y(x) = \propto \ \delta(x - y)$$

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1. We have to do, again, with a non-square integrable eigenfunction.

2. This means that, strictly speaking, also for the position operator $\hat{x}$ there are **no eigenvectors** in the Hilbert space of the physical states.

3. In other words, **we cannot realize a physical state such that a position measurement on it gives with certainty a well defined real number !**

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Continuous spectrum

Similarly to what happens for the momentum generalized eigenvectors, the generalized eigenfunctions

$$\left\{ g_y(x) = \delta(x - y); \;\; y \in R \right\}$$

form a ***complete, orthonormal set***.

Continuous spectrum

In fact, for any (square-integrable) continuous function $\psi(x)$, it exists always the scalar product

$$\bar{\psi}(y) \equiv < g_y|\psi >= \int dx\, g_y^*(x) \cdot \psi(x) = $$
$$= \int dx\ \delta(x-y)\psi(x) = \psi(y)$$

and, clearly, we have

$$\psi(x) = \int dy\, \bar{\psi}(y) g_y(x) = \int dy\, \psi(y)\delta(x-y)$$

Continuous spectrum

Moreover, the generalized position eigenfunctions $g_y(x) = \delta(y - x)$ satisfy the Dirac generalized orthonormality condition.

In fact

$$\int dx\, g_y^*(x) g_z(x) = \int dx\, \delta(y - x) g_z(x) =$$
$$= \; g_z(y) = \delta(y - z)$$

Continuous spectrum

The two examples considered so far, momentum and position, lead us to the following conclusion.

If the spectrum $\mathcal{S} \subset R$ of the eigenvalues s of an observable $\hat{Q}$ is continuous, the corresponding **eigenvectors $\{\mathbf{e}(s); s \in \mathcal{S}\}$, although they do not belong to the Hilbert space, they can be used as elements of a generalized basis.**

Continuous spectrum

In fact, **these generalized eigenvectors $\mathbf{e}(s)$ can be made orthonormal in the Dirac sense**:

$$< \mathbf{e}(s)|\mathbf{e}(t) >= \delta(s-t)$$

and any normalized vector $\mathbf{v} \in \mathcal{H}$ can be written as

$$\mathbf{v} = \int ds\, \phi(s)\, \mathbf{e}(s)$$

where the complex function $\phi(s)$ is given by

$$\phi(s) \equiv < \mathbf{e}(s)|\mathbf{v} >$$

Generalized statistical interpretation

Concerning the expectation value of $\hat{Q}$, we have

$$< Q > = < \mathbf{v}|\hat{Q}\,\mathbf{v} > =$$
$$= \int ds\,dt < \phi(s)\mathbf{e}(s)|\hat{Q}\,\phi(t)\mathbf{e}(t) > =$$
$$= \int ds\,dt\,\phi(s)^*\,\phi(t) < \mathbf{e}(s)|\hat{Q}\,\mathbf{e}(t) > =$$
$$= \int ds\,dt\,\phi(s)^*\,\phi(t)\,t < \mathbf{e}(s)|\mathbf{e}(t) > =$$
$$= \int ds\,dt\,\phi(s)^*\,\phi(t)\,t\,\delta(t-s) = \int ds\;|\phi(s)|^2\,s$$

and $|\phi(s)|^2$ represents the probability density function to obtain a value between s and $s + ds$ when measuring $\hat{Q}$ on $\mathbf{v}$.

Generalized statistical interpretation

1. The generalized statistical intepretation agrees, of course, with what we have called "the Copenaghen interpretation", for which, $|\Psi(x)|^2$ is the p.d.f. to find the particle between x and $x + dx$.

2. In fact, the *old* wave function $\Psi(x)$ is nothing but what we are calling, now, $\phi(x)$ when the physical vector state is Ψ and the observable $\hat{Q}$ is the position $\hat{x}$

$$\begin{aligned}\phi(x) &= < g_x|\Psi >= \int dy\, \delta(x-y)^*\, \Psi(y) = \\ &= \Psi(x)\end{aligned}$$

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Generalized statistical interpretation

1. Concerning the momentum operator $\hat{p}$, we have already said that its generalized orthonormal eigenfunctions are

$$\mathbf{e}(p) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar} \quad with \quad p \in R$$

and one has

$$\phi(p) \;=\; < \mathbf{e}(p)|\Psi >= \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-ipx/\hbar}\; \Psi(x)$$

2. The quantity $|\phi(p)|^2$ is the p.d.f. to measure a momentum between p and $p + dp$ and $\phi(p)$ can be seen as the wave-function of the state Ψ in the momentum space.

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Exercise N.7

Exercise

In a three dimensional Hilbert space $\mathcal{H}$, the spectrum of the observable Q is $\{-1, 0, +1\}$. Let $\{\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+\}$ be the orthonormal basis made by the eigenvectors of Q

$$Q\mathbf{e}_- = \mathbf{e}_-; \quad Q\mathbf{e}_0 = \mathbf{0}; \quad Q\mathbf{e}_+ = -\mathbf{e}_+$$

- If $\mathbf{v} = \alpha\mathbf{e}_- + \beta\mathbf{e}_0 + \gamma\mathbf{e}_+$ is a generic vector of $\mathcal{H}$, which is the expectation value of Q on the physical state described by $\mathbf{v}$?
- Write the condition on the coefficients α, β, γ for which the expectation value of Q is zero.
- Is it possible to find a basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ for which $< \mathbf{f}_i | Q\mathbf{f}_i >= 0$? Explain.

Compatible and incompatible observables

1. Up to now, we have considered only eigenvalues/eigenvectors of a single observable $\hat{Q}$.

2. Before continuing, we want to remember that the eigenvectors of $\hat{Q}$, **corresponding to the same eigenvalue** q, form a **linear subspace** since any linear combination of these eigenvectors is still a $\hat{Q}$ eigenvector for the eigenvalue q.

3. Let us call $\mathcal{V}$ this linear space. By definition

$$\mathbf{v} \in \mathcal{V} \Leftrightarrow \hat{Q}\mathbf{v} = q\,\mathbf{v}$$

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Compatible and incompatible observables

1. Let us consider, now, two observables $\hat{Q}_1$ and $\hat{Q}_2$ and let us assume that $[\hat{Q}_1, \hat{Q}_2] = 0$. Let $\mathcal{V}_1$ be the eigenspace of $\hat{Q}_1$ corresponding to the eigenvalue q_1 and let $\mathbf{v} \in \mathcal{V}_1$. Let us consider the vector $\hat{Q}_2\mathbf{v}$: we have

$$\hat{Q}_1 \left(\hat{Q}_2\mathbf{v}\right) = \hat{Q}_2 \left(\hat{Q}_1\mathbf{v}\right) = q_1 \, \hat{Q}_2\mathbf{v}$$

   which means that also $\hat{Q}_2\mathbf{v} \in \mathcal{V}_1$.

2. The subspace $\mathcal{V}_1$ is, therefore, **invariant** also under $\hat{Q}_2$. But $\hat{Q}_2$ is hermitian and, therefore, we can find an orthonormal basis of $\mathcal{V}_1$ made by eigenvectors of $\hat{Q}_2$: these vectors are **simultaneously** eigenvectors of $\hat{Q}_1$ and $\hat{Q}_2$.

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Compatible and incompatible observables

1. The procedure can be repeated for all the eigenvalues of $\hat{Q}_1$ and the conclusion is that, if $\hat{Q}_1$ and $\hat{Q}_2$ commute, we can find an orthonormal basis of the whole space which is made by **simultaneous eigenvectors of both the operators.**
2. We say that $\hat{Q}_1$ and $\hat{Q}_2$ are **compatible**.
3. This means that there are **physical states in which both observables are determinate.**
4. If $[\hat{Q}_1, \hat{Q}_2] \neq 0$, a basis of common eigenvectors cannot exist and the observables $\hat{Q}_1$ and $\hat{Q}_2$ are **incompatible** (f.i., $\hat{x}$ and $\hat{p}$).

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The uncertainty principle revisited

1. Let us come, now, to reconsider the **uncertainty principle**.
2. We will show that, in the mathematical framework developed so far, it **is not**, really, **a principle**, but, it **is**, in fact, **a theorem** ...
3. Let $\hat{A}$ and $\hat{B}$ two generic observables (hermitian operators) and let us define the two following non-hermitian operators

$$\hat{C} \equiv \hat{A} + i\alpha\,\hat{B} \quad \Leftrightarrow \quad \hat{C}^\dagger = \hat{A} - i\alpha\,\hat{B}$$

where α is a generic real number. Then

$$\hat{C}^\dagger \hat{C} = \left(\hat{A} - i\alpha\,\hat{B}\right)\left(\hat{A} + i\alpha\,\hat{B}\right) =$$
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2. Since the vectors $\{\mathbf{e}_i\}$ form a basis, we will have

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2. Therefore, remembering the definition of σ^2 in terms of the variance and the average, we can conclude[1] that, for any real number α

$$\sigma_A^2 + \alpha^2 \sigma_B^2 + i\alpha < [\hat{A}, \hat{B}] > \geq 0$$

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1. The second degree equation in the real variable α that we have obtained, to be always non-negative, must have the discriminant smaller or equal to zero:

$$\left(i < [\hat{A}, \hat{B}] >\right)^2 - 4\sigma_A^2\sigma_B^2 \leq 0$$

$$\Rightarrow 4\sigma_A^2\sigma_B^2 \geq \left(i < [\hat{A}, \hat{B}] >\right)^2$$

2. If we consider, now, for instance, the observables $\hat{x}$ and $\hat{p}$, since $[\hat{x}, \hat{p}] = i\hbar$, we have

$$4\sigma_x^2\sigma_p^2 \geq \left(i^2\hbar\right)^2 \quad \Rightarrow \quad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

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