QUANTUM MECHANICS Lecture 16

The Hilbert space

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D. J. Griffiths: paragraphs 3.1

Hilbert space

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2. Some of these properties are **accidentals** (such as, f.i., the constant spacing between consecutive energy levels for the harmonic oscillator); but some of them are of **general nature** (such as the uncertainty principle, the orthogonality of the stationary states ...).

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Hilbert space

1. We have learned that the physical states are described by **square integrable** w.f.
2. We have also learned that the physical observables are described by ***operators***: like $x \to \hat{x} \equiv x\cdot;$ $p \to \hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$ such that

$$< x > = \int dx\, \Psi^* \, (\hat{x}\, \Psi) = \int dx\, \Psi^* \, x\, \Psi$$

$$< p > = \int dx\, \Psi^* \, (\hat{p}\, \Psi) = -i\hbar \int dx\, \Psi^* \, \frac{\partial \Psi}{\partial x}$$

3. And in general, for any physical observable $Q(x,p) \to \hat{Q} \equiv Q(x, -i\hbar\frac{\partial}{\partial x})$ we have indeed

$$< Q >= \int dx\, \Psi^* \, (\hat{Q}\, \Psi)$$

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1. The most natural mathematical structure able to describe properly these fundamental aspects is the one of the **complex vector spaces.**

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The commutative group structure requires that

- if $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathcal{H}$, then $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}\in\mathcal{H}$
- there exists the null vector $\mathbf{0}$, such that $\mathbf{a}+\mathbf{0}=\mathbf{a}$ for any $\mathbf{a}\in\mathcal{H}$
- the sum is associative $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$
- every element $\mathbf{a}$ of $\mathcal{H}$ has its own (unique) *opposite* $\bar{\mathbf{a}}$, such that $\mathbf{a}+\bar{\mathbf{a}}=\mathbf{0}$

Hilbert space

The multiplication by **scalars** must be such that

- if $\alpha \in C$ and $\mathbf{a} \in \mathcal{H} \Rightarrow \alpha\,\mathbf{a} \in \mathcal{H}$
- $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\,\mathbf{a} + \alpha\,\mathbf{b}$
- $0\,\mathbf{a} = \mathbf{0}$
- $(-1)\mathbf{a} = \overline{\mathbf{a}} \equiv -\mathbf{a}$

Hilbert space

1. But this linear structure **is not enough ....**

2. We have also seen that the **statistical interpretation** has to do with the integral

$$\int dx\, \Psi(x,t)^* \; \Psi(x,t) \equiv \int dx\, |\Psi(x,t)|^2$$

3. In order **to deal with this aspect**, the most suitable structure to set up the theory of QM is the **Hilbert space.**

4. A Hilbert space is a **vector space on the complex field** in which it is defined an *inner scalar product*; **not to be confused with the *product by complex scalars* mentioned before ...**

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Hilbert space

1. The inner *scalar product* $<,>$ is such that, for any $\mathbf{a}$, $\mathbf{b}, \mathbf{c} \in \mathcal{H}$ and $\alpha \in C$
   - $< \mathbf{a}|\mathbf{b} > \in C$
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   - $< \mathbf{a}|\mathbf{b} > = < \mathbf{b}|\mathbf{a} >^*$
   - $< \mathbf{a}|\mathbf{a} > \equiv |\mathbf{a}|^2 \geq 0; \quad = 0 \;\; iff \;\; \mathbf{a} = \mathbf{0}$
   - $< \mathbf{a}|\alpha\, \mathbf{b} > = \alpha < \mathbf{a}|\mathbf{b} > \;\; \Rightarrow$
   - $\Rightarrow \;\; < \alpha\mathbf{a}|\mathbf{b} > = \alpha^* < \mathbf{a}|\mathbf{b} >$

2. One of the most relevant consequences of the above properties is the **Schwarz inequality**, for which, in general, we have

$$| < \mathbf{a}|\mathbf{b} > |^2 \;\; \leq \;\; < \mathbf{a}|\mathbf{a} > < \mathbf{b}|\mathbf{b} > \equiv |\mathbf{a}|^2 \cdot |\mathbf{b}|^2$$

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QM axiom:

1. The states of any physical system are described by vectors of a suitable Hilbert space.
2. If two vectors are proportional: $w = \alpha\, v$ they describe the same physical state.

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Basis in a Hilbert space

The vectors $\{\mathbf{e_i}\}$ form a basis for the Hilbert space iff

- they are linearly independent, i.e.

$$\sum_i \alpha_i \, \mathbf{e_i} \equiv \alpha_i \, \mathbf{e_i} = \mathbf{0} \iff \forall i \; : \alpha_i = 0$$

- every vector $\mathbf{v}$ belonging to the Hilbert space can be written as a linear combination of the elements of the basis:
  $\mathbf{v} = \gamma_i \, \mathbf{e_i}$
  where the γ_i are suitable complex numbers that, due to the linear independence property of the $\{\mathbf{e_i}\}$, **are unique.**

Hilbert space

1. The **number of elements** of a basis is a **characteristic of the Hilbert space,** which can be **finite** or **numerable.**

2. It can be shown that we can always choose a basis which is *orthonormal*, which means that $< \mathbf{e_i} | \mathbf{e_j} >= \delta_{ij}$, where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 0$ when $i \neq j$; $\delta_{ij} = 1$ when $i = j$).

3. When the basis is orthonormal, the coefficients γ_j are simply given by the scalar product of the vector with the elements of the basis:

$$\gamma_j =< \mathbf{e_j} | \mathbf{v} > \quad \Rightarrow \quad \mathbf{v} = \sum_j < \mathbf{e_j} | \mathbf{v} > \mathbf{e_j}$$

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1. However, a Hilbert space may (it will !) have various different orthonormal bases.
2. Let $\{\mathbf{e_j}\}$ and $\{\mathbf{f_k}\}$ be two of them.
3. Since $\{\mathbf{e_j}\}$ is a basis

$$\forall i : \mathbf{f_i} = \sum_j A_{ji} \mathbf{e_j} \equiv A_{ji} \mathbf{e_j}$$

and because also $\{\mathbf{f_k}\}$ is a basis

$$\forall m : \mathbf{e_m} = B_{km} \mathbf{f_k}$$

Therefore

$$\mathbf{e_m} = B_{km} A_{jk} \mathbf{e_j} \quad \Leftrightarrow \quad (AB)_{jm} = \delta_{jm}$$

which means that the matrices A and B are such that $B = A^{-1} \Leftrightarrow A = B^{-1}$.

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