QUANTUM MECHANICS Lecture 15

Midterm execise solution
The finite barrier

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D. J. Griffiths: paragraph 2.6

Exercise

The state of a particle in a harmonic potential is described, at $t = 0$, by the wave function

$$\Psi(x, 0) = A\,[\psi_0(x) - 2i\,\psi_1(x) + 2\,\psi_2(x)]$$

where the $\psi_n(x)$ are the normalized solutions of the time independent Schrödinger equation for the energies $E_n = (n + 1/2)\hbar\omega$.
Determine

- the normalization constant A;
- the time dependent wave function $\Psi(x, t)$;
- the expectation value $< H >$ of the total energy on $\Psi(x, t)$;
- the probability $P(t)$ to obtain the value E_2 and E_3 as a result of an energy measurement on the state described by $\Psi(x, t)$.

Solution

Normalization

We have

$$\begin{aligned} 1 &= \int dx\, \Psi^*(x,0)\Psi(x,0) = \\ &= A^2 \int dx\, (\psi_0^* + 2i\psi_1^* + 2\psi_2^*)(\psi_0 - 2i\psi_1 + 2\psi_2) \end{aligned}$$

The orthonormality of the ψ_n implies that

$$\int dx\, \psi_n^*(x)\psi_m(x) = \delta_{nm}$$

therefore, for the normalization condition, we obtain

$$1 = A^2(1+4+4) \quad \Rightarrow \quad A = \frac{1}{3}$$

Solution

Time dependence

$$
\begin{aligned}
\Psi(x,t) &= A\left[\psi_0(x)e^{-i\omega t/2} - 2i\psi_1(x)e^{-3i\omega t/2} + \right.\\
&\left. +2\psi_2(x)e^{-5i\omega t/2}\right] = \\
&= \frac{1}{3}e^{-i\omega t/2}\left[\psi_0(x) - 2i\psi_1 e^{-i\omega t} + \right.\\
&\left. +2\psi_2(x)e^{-2i\omega t}\right]
\end{aligned}
$$

Solution

Energy expectation value

Since $< H >$ is time independent, we can evaluate it at $t = 0$. We have

$$
\begin{aligned}
< H > &= \int dx\, \Psi^*(x,0)\, H\, \Psi(x,0) = \\
&= A^2 \int dx\, [\psi_0^* + 2i\psi_1^* + 2\psi_2^*] \cdot \\
\cdot &\quad [E_0\psi_0 - 2iE_1\psi_1 + 2E_2\psi_2] = \\
&= \frac{1}{9}\,(E_0 + 4E_1 + 4E_2) = \\
&= \frac{\hbar\omega}{18}(1 + 3\cdot 4 + 5 \cdot 4) = \frac{33}{18}\hbar\omega = \frac{11}{6}\hbar\omega
\end{aligned}
$$

where we have used again the ψ_n orthonormality properties.

Solution

Probability of obtaining E_n

This probability is given by

$$P_n = |c_n|^2$$

where

$$c_n = \int dx\, \psi_n(x)^* \, \Psi(x, 0)$$

In our case we have

$$\begin{aligned} c_2 &= \frac{2}{3} &\Rightarrow\quad P_1 = \frac{4}{9} \\ c_3 &= 0 &\Rightarrow\quad P_2 = 0 \end{aligned}$$

§§§§§§§

Finite potential barrier

1. Let us come, now, to the argument of the **lecture of today.**
2. We have studied the finite square well.
   It is , now, interesting to consider the similar problem of the **square potential barrier.**
3. In this case, the potential is the following

$$
\begin{aligned}
V(x) &= 0 \quad\;\; if\ |x| > a \\
V(x) &= +V_0\ if\ |x| \leq a
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   with $V_0 > 0$.
4. Similarly to what happens for the free particle, it is easy to convince ourselves that there are **no stationary solutions if E < 0.**

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Finite potential barrier

1. If $E > 0$ the stationary solutions represent **only scattering states**.

2. If $\mathbf{E} > \mathbf{V_0}$, we have again

$$
\begin{aligned}
x < -a &\quad : \quad \psi(x) = A\,e^{ikx} + B\,e^{-ikx} \\
|x| \le a &\quad : \quad \psi(x) = C\,e^{irx} + D\,e^{-irx} \\
x > a &\quad : \quad \psi(x) = F\,e^{ikx} + G\,e^{-ikx}
\end{aligned}
$$

where, now, $k \equiv \frac{\sqrt{2mE}}{\hbar}$ and $r \equiv \frac{\sqrt{2m(E-V_0)}}{\hbar}$.

3. The coefficients A, B, C, D, F, G, in terms of k and r, are formally the same as for the finite square well, because the algebra that defines them is exactly the same.

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Finite potential barrier

1. The transmission coefficient $\mathcal{T}$, for a left-to-right scattering process, is, therefore, still given by the "old" expression

$$\mathcal{T} = \frac{4r^2k^2}{4r^2k^2 + (r^2 - k^2)^2 sin^2(2ra)}$$

2. which, in terms of the energy E and V_0, now becomes ($V_0 \to -V_0$)

$$\mathcal{T} = \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 sin^2(2ra)}$$

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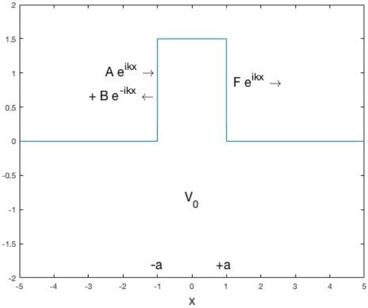
Finite potential barrier

Let us assume, now, that **$0 < E < V_0$**.



Finite potential barrier

The wave function in the potential region is now a linear combination of two **real** exponentials

$$|x| \leq a \quad : \quad \psi(x) = C\, e^{-\hat{r}x} + D\, e^{\hat{r}x}$$

where $\hat{r} \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar}$.

Finite potential barrier

1. We can use directly the algebraic relations concerning the continuity of ψ and $\psi^{'}$, already established, if we identify

$$r \to -i\,\hat{r} = -i\,\frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

2. which implies the substitutions

$$\begin{aligned} i\,sin(2ra) &\to sh(2\hat{r}a) \\ cos(2ra) &\to ch(2\hat{r}a) \end{aligned}$$

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Finite potential barrier

1. For a left-to-right scattering, now we have

$$F = A\, e^{-ika}\, \frac{-2i\, k\hat{r}}{-2i\, k\hat{r}\; ch(2\hat{r}a) - (-\hat{r}^2 + k^2) sh(2\hat{r}a)} =$$

$$= A\, e^{-ika}\, \frac{2\, k\hat{r}}{2\, k\hat{r}\; ch(2\hat{r}a) - i(k^2 - \hat{r}^2) sh(2\hat{r}a)}$$

2. and, therefore, the transmission coefficient is given by

$$\mathcal{T} = \left|\frac{F}{A}\right|^2 = \frac{4\hat{r}^2 k^2}{4\hat{r}^2 k^2 ch^2(2\hat{r}a) + (k^2 - \hat{r}^2)^2 sh^2(2\hat{r}a)}$$

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Finite potential barrier

Since $ch^2x - sh^2x = 1$, we have

$$\mathcal{T} = \left|\frac{F}{A}\right|^2 =$$

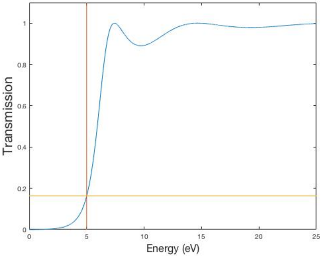
$$= \frac{4\hat{r}^2k^2}{4\hat{r}^2k^2[1 + sh^2(2\hat{r}a)] + (k^2 - \hat{r}^2)^2 sh^2(2\hat{r}a)} =$$

$$= \frac{4\hat{r}^2k^2}{4\hat{r}^2k^2 + (k^2 + \hat{r}^2)^2 sh^2(2\hat{r}a)} =$$

$$= \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 sh^2(2\hat{r}a)}$$

which shows that, **also when the energy E is smaller than the potential V_0**, there is a **non-zero transmission** from the barrier: it is the so-called ***tunnel effect***.

Finite potential barrier



The transmission shown in the picture refers to an electron which scatters against a potential well of depth $V_0 = +5eV$ and width $2a = 0.4nm$.

The transmission $\mathcal{T}$ is continuous in $E = V_0$ and it is easily evaluated to be

$$\mathcal{T}(V_0) = \frac{1}{1 + \frac{2mV_0a^2}{\hbar^2}}$$

which is the maximum transmission possible for the tunnel effect ($E \leq V_0$).