Exercise N.7

Exercise

In a three dimensional Hilbert space $\mathcal{H}$, the spectrum of the observable Q is $\{-1, 0, +1\}$. Let $\{\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+\}$ be the orthonormal basis made by the eigenvectors of Q

$$Q\mathbf{e}_- = -\mathbf{e}_-; \quad Q\mathbf{e}_0 = \mathbf{0}; \quad Q\mathbf{e}_+ = \mathbf{e}_+$$

- If $\mathbf{v} = \alpha\mathbf{e}_- + \beta\mathbf{e}_0 + \gamma\mathbf{e}_+$ is a generic vector of $\mathcal{H}$, which is the expectation value of Q on the physical state described by $\mathbf{v}$?
- Write the condition on the coefficients α, β, γ for which the expectation value of Q is zero.
- Is it possible to find a basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ for which $< \mathbf{f}_i|Q\mathbf{f}_i >= 0$? Explain.

Solution

1. The expectation value of Q on the generic vector $\mathbf{v} = \alpha \mathbf{e}_- + \beta \mathbf{e}_0 + \gamma \mathbf{e}_+$ is given by the scalar product

$$\begin{aligned} < \mathbf{v}|Q\mathbf{v} > &= < \alpha \mathbf{e}_- + \beta \mathbf{e}_0 + \gamma \mathbf{e}_+| - \alpha \mathbf{e}_- + \gamma \mathbf{e}_+ >= \\ &= -|\alpha|^2 + |\gamma|^2 \end{aligned}$$

where we have used the fact that the vectors $\{\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+\}$ form an orthonormal basis and that they are eigenvectors of Q for the eigenvalues -1, 0, $+1$, respectively.

2. Clearly, the condition on α, β, γ for which $< \mathbf{v}|Q\mathbf{v} >= 0$ is that $|\alpha|^2 = |\gamma|^2$.

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Solution

1. A basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ such that $< \mathbf{f}_i|Q\mathbf{f}_i >= 0$ **can be found** (in a non-unique way ...).
2. As an example, we can take $\mathbf{f}_1 = \mathbf{e}_0$ and since $\mathbf{e}_0$ is an eigenvector of Q for the eigenvalue 0, clearly
$$< \mathbf{f}_1|Q\mathbf{f}_1 >=< \mathbf{e}_0|Q\mathbf{e}_0 >= 0$$
Then, we can complete the basis, for instance, with the two orthogonal vectors
$$\mathbf{f}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_- + \mathbf{e}_+) \qquad \mathbf{f}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_- - \mathbf{e}_+)$$
which satisfy the condition found above for a null Q expectation value.
3. The above basis is orthonormal.

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