Exercise N.4

Exercise

A harmonic oscillator is described, at $t = 0$, by the w.f. $\Psi(x, 0) = A\left(3\psi_0(x) + 4\psi_1(x)\right)$

a) find the normalization constant A;

b) determine the p.d.f $|\psi(x, t)|^2$;

c) find $< x >$, $< p >$ and $< E >$ as functions of time.

Solution

To solve the problem it is useful to use the relations

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right)$$

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(a_+ - a_-\right)$$

$$a_+\,\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$a_-\,\psi_n = \sqrt{n}\,\psi_{n-1}$$

Solution

Normalization constant A

The w.f. at $t = 0$ is

$$\Psi(x,0) = A\Big(3\psi_0(x) + 4\psi_1(x)\Big)$$

Since ψ_0 and ψ_1 are orthonormal, we have

$$\int dx\,|\Psi(x,0|^2 = |A|^2(9+16) \;\Rightarrow\; A = \frac{1}{5}$$

and the normalization, as we know, is time-independent.

Solution

P.d.f. $|\psi(x,t)|^2$

$$\Psi(x,t) = \frac{1}{5}\left\{3\psi_0(x)e^{-\frac{i}{2}\omega t} + 4\psi_1(x)e^{-\frac{3i}{2}\omega t}\right\}$$
$$= \frac{1}{5}e^{-\frac{i}{2}\omega t}\left\{3\psi_0(x) + 4\psi_1(x)e^{-i\omega t}\right\}$$

$$|\Psi(x,t)|^2 = \frac{1}{25}\left\{3\psi_0(x) + 4\psi_1(x)e^{i\omega t}\right\} \cdot$$
$$\cdot \quad \left\{3\psi_0(x) + 4\psi_1(x)e^{-i\omega t}\right\} =$$
$$= \frac{1}{25}\left\{9\psi_0^2 + 12\psi_0\psi_1\left(e^{-i\omega t} + e^{i\omega t}\right) + 16\psi_1^2\right\} =$$
$$= \frac{1}{25}\left\{9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 cos\omega t\right\} =$$
$$= \frac{1}{25}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\xi^2}\left\{9 + 32\xi^2 + 24\xi\, cos\omega t\right\}$$

Solution

$< x >$ and <p>

Since $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right)$ and

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}; \quad a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

we have

$$\begin{aligned} < x > &= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{25}\int dx\left[3\psi_0(x) + 4\psi_1(x)e^{-i\omega t}\right]^* \cdot \\ &\quad \cdot (a_+ + a_-)\left[3\psi_0(x) + 4\psi_1(x)e^{-i\omega t}\right] = \\ &= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{25}\int dx\left[3\psi_0(x) + 4\psi_1(x)e^{i\omega t}\right] \cdot \\ &\quad \cdot \left[3\psi_1(x) + 4\sqrt{2}\psi_2(x)e^{-i\omega t} + 4\psi_0(x)e^{-i\omega t}\right] \end{aligned}$$

Solution

1. But the ψ_n are orthonormal, therefore

$$< x > \; = \; \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{25}\left(12e^{-i\omega t} + 12e^{i\omega t}\right) =$$
$$= \; \sqrt{\frac{\hbar}{2m\omega}}\frac{24}{25}\, cos\omega\, t$$

2. Concerning $< p >$, with the same procedure we obtain

$$< p > \; = \; i\sqrt{\frac{m\hbar\omega}{2}}\frac{1}{25}\left(-12e^{-i\omega t} + 12e^{i\omega t}\right) =$$
$$= \; -\sqrt{\frac{m\hbar\omega}{2}}\frac{24}{25} sin\omega t \equiv m\frac{d}{dt} < x >$$

Solution

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Solution

<H>

We have

$$
\begin{aligned}
< H > &= \frac{1}{25}\int dx\,\left[3\psi_0(x)+4\psi_1(x)e^{-i\omega t}\right]^* \cdot \\
&\cdot \quad \widehat{H}\,\left[3\psi_0(x)+4\psi_1(x)e^{-i\omega t}\right] = \\
&= \frac{1}{25}\int dx\,\left[3\psi_0(x)+4\psi_1(x)e^{i\omega t}\right]^* \cdot \\
&\cdot \quad \left[\frac{1}{2}\hbar\omega\;3\psi_0(x)+\frac{3}{2}\hbar\omega\;4\psi_1(x)e^{-i\omega t}\right] = \\
&= \frac{1}{25}\left(9\,\frac{1}{2}\hbar\omega+16\,\frac{3}{2}\hbar\omega\right)=\frac{57}{50}\hbar\omega
\end{aligned}
$$

and an energy measurement will give **only** $\frac{1}{2}\hbar\omega$ ($P=\frac{9}{25}$) or $\frac{3}{2}\hbar\omega$ ($P=\frac{16}{25}$).