Exercise N.2

Exercise

The physical state of a particle is represented at $t = 0$ by the following wave function

$$\begin{aligned} \Psi(x,0) &= A(a^2 - x^2) \; for \; |x| \leq a \\ \psi(x,0) &= 0 \; for \; |x| > a \end{aligned}$$

where A and a are positive (real) constants.

a) determine the normalization constant A;
b) calculate the expectation value of x;
c) calculate the expectation value of p;
d) find σ_x;
e) find σ_p;
f) evaluate the product $\sigma_x \, \sigma_p$.

Solution

Normalization:

$$
\begin{aligned}
1 &= A^2 \int_{-a}^{+a} dx\,(a^2 - x^2)^2 = 2A^2 \int_0^a dx\,(a^2 - x^2)^2 = \\
&= 2A^2 \int_0^a dx\,(a^4 + x^4 - 2a^2x^2) = \\
&= 2A^2(a^4 a + \frac{1}{5}a^5 - 2a^2\frac{1}{3}a^3) = \\
&= 2A^2a^5(1 + \frac{1}{5} - \frac{2}{3}) = 2A^2a^5\frac{8}{15} \Rightarrow \\
&\Rightarrow A^2 = \frac{15}{16}a^{-5} \\
&\Rightarrow A = \frac{\sqrt{15}}{4}a^{-5/2}
\end{aligned}
$$

Solution

- Expectation value $< x >$:

$$< x > \; = \; A^2 \int_{-a}^{+a} dx \, (a^2 - x^2)^2 \, x = 0$$

because the integrand is an **odd** function.

- Expectation value $< p >$:

$$< p > = -i\hbar \, A^2 \int_{-a}^{+a} dx \, (a^2 - x^2) \left(\frac{d}{dx} (a^2 - x^2) \right) =$$

$$= -i\hbar \, A^2 \int_{-a}^{+a} dx \, (a^2 - x^2) \, (-2x) = 0$$

because the integrand is, again, odd.

Solution

- Expectation value $< x >$:

$$< x > \; = \; A^2 \int_{-a}^{+a} dx \, (a^2 - x^2)^2 \, x = 0$$

because the integrand is an **odd** function.

- Expectation value $< p >$:

$$< p > = -i\hbar \, A^2 \int_{-a}^{+a} dx \, (a^2 - x^2) \left(\frac{d}{dx} (a^2 - x^2) \right) =$$

$$= -i\hbar \, A^2 \int_{-a}^{+a} dx \, (a^2 - x^2) \, (-2x) = 0$$

because the integrand is, again, odd.

Solution

σ_x:
since $< x >= 0$, $\sigma_x^2 =< x^2 >$

$$
\begin{aligned}
< x^2 > &= A^2 \int_{-a}^{+a} dx\,(a^2 - x^2)^2 x^2 = \\
&= 2A^2 \int_0^a dx\,(a^4 + x^4 - 2a^2x^2)x^2 = \\
&= 2A^2(a^4 \frac{1}{3}a^3 + \frac{1}{7}a^7 - 2a^2\frac{1}{5}a^5) = \\
&= 2A^2a^7(\frac{1}{3} + \frac{1}{7} - \frac{2}{5}) = A^2a^7\frac{16}{105} = \\
&= \frac{15}{16}a^{-5}a^7\frac{16}{105} = \frac{a^2}{7} \Rightarrow \\
\Rightarrow & \quad \sigma_x = \frac{a}{\sqrt{7}}
\end{aligned}
$$

Solution

σ_p:
since $< p >= 0$, $\sigma_p^2 =< p^2 >$

$$< p^2 > = -\hbar^2 A^2 \int_{-a}^{+a} dx\,(a^2 - x^2)\left(\frac{d^2}{dx^2}(a^2 - x^2)\right) =$$

$$= -2\hbar^2 A^2 \int_0^a dx\,(a^2 - x^2)(-2) =$$

$$= 4A^2\hbar^2(a^3 - \frac{1}{3}a^3) = 4\frac{15}{16}a^{-5}\hbar^2 a^3 \frac{2}{3} =$$

$$= \frac{5}{2}a^{-2}\hbar^2 \;\Rightarrow\; \sigma_p = a^{-1}\hbar\sqrt{\frac{5}{2}}$$

$$\Rightarrow \sigma_x \cdot \sigma_p = a\frac{1}{\sqrt{7}} \cdot a^{-1}\hbar\sqrt{\frac{5}{2}} = \hbar\sqrt{\frac{5}{14}}$$

$$\approx 0.598\,\hbar \;\left(> \frac{\hbar}{2} \ldots\right)$$