Exercise N.1

Exercise

Assume a wave function

$$\Psi(x,t) = A\, e^{-\lambda\,|x|}\; e^{-i\omega\, t}$$

where A, λ and ω are positive real constants.

i) Normalize Ψ;
ii) Evaluate $< x >$, $< x^2 >$ and σ;
iii) Calculate the probability to find the particle outside the interval $< x > \pm\sigma$.

Solution

1. Normalization:

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2 = A^2 \int_{-\infty}^{+\infty} dx\, e^{-2\lambda|x|} = \\ &= 2A^2 \int_0^{+\infty} dx\, e^{-2\lambda x} = 2A^2 \frac{e^{-2\lambda x}}{-2\lambda}\Big|_0^{+\infty} = \\ &= \frac{A^2}{\lambda} \quad \Rightarrow \quad A = \sqrt{\lambda} \end{aligned}$$

2. The expectation value $< x >= 0$, since $|\Psi|^2$ is an even function of x.

Solution

1. Normalization:

$$
\begin{aligned}
1 &= \int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2 = A^2 \int_{-\infty}^{+\infty} dx\, e^{-2\lambda|x|} = \\
&= 2A^2 \int_{0}^{+\infty} dx\, e^{-2\lambda x} = 2A^2 \left.\frac{e^{-2\lambda x}}{-2\lambda}\right|_0^{+\infty} = \\
&= \frac{A^2}{\lambda} \quad \Rightarrow \quad A = \sqrt{\lambda}
\end{aligned}
$$

2. The expectation value $< x >= 0$, since $|\Psi|^2$ is an even function of x.

Solution

1. Expectation value $< x^2 >$ and σ

$$
\begin{aligned}
< x^2 > &= 2\lambda \int_0^{+\infty} dx\, e^{-2\lambda x} \cdot x^2 = \\
&= -\int_0^{+\infty} dx\, x^2 \left(\frac{d}{dx} e^{-2\lambda x} \right) = \\
&= \left(-x^2 e^{-2\lambda x} \right)\Big|_0^{+\infty} + \int_0^{+\infty} dx\, e^{-2\lambda x} \cdot 2x = \\
&= \frac{-1}{2\lambda} \int_0^{+\infty} dx \left(\frac{d}{dx} e^{-2\lambda x} \right) 2x = \\
&= \left(-\frac{x}{\lambda} e^{-2\lambda x} \right)\Bigg|_0^{+\infty} + \frac{1}{\lambda} \int_0^{+\infty} dx\, e^{-2\lambda x} = \\
&= \frac{-1}{2\lambda^2}\, e^{-2\lambda x}\Big|_0^{+\infty} = \frac{1}{2\lambda^2} \Rightarrow \sigma = \frac{1}{\lambda\sqrt{2}}
\end{aligned}
$$

Solution

1. Probability P to find the particle outside the interval $< x > \pm\sigma \equiv \pm\sigma = \pm\frac{1}{\lambda\sqrt{2}}$:

$$
\begin{aligned}
P &= 2\lambda \int_{\sigma}^{+\infty} dx\, e^{-2\lambda x} = 2\lambda \frac{1}{-2\lambda}\, e^{-2\lambda x}\Big|_{\sigma}^{+\infty} = \\
&= e^{-2\lambda\sigma} = e^{-\sqrt{2}} \approx 0.243
\end{aligned}
$$

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