QUANTUM MECHANICS Appendix 4

Scattering coefficients for the finite square well

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Appendix4: Calculation of A,B,C,D,F for the finite potential well

Let us evaluate the coefficients A, B, C, D, F for the scattering states associated to the finite potential well. We have seen that

$$x = -a \quad : \quad Ae^{-ika} + Be^{ika} = Ce^{-ira} + De^{ira}$$
$$ikAe^{-ika} - ikBe^{ika} = irCe^{-ira} - irDe^{ira}$$

$$x = +a \quad : \quad Ce^{ira} + De^{-ira} = Fe^{ika}$$
$$irCe^{ira} - irBe^{-ira} = ikFe^{ika}$$

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Using the two continuity conditions for $x = +a$, by multiplying the first equation for k and comparing the two, we obtain

$$
\begin{aligned}
& kCe^{ira} + kDe^{-ira} = rCe^{ira} - rDe^{-ira} \\
\Rightarrow \quad & De^{-ira}(r+k) = Ce^{ira}(r-k)
\end{aligned}
$$

$$
\Rightarrow \quad D = C\, e^{2ira}\, \frac{r-k}{r+k}
$$

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and therefore

$$\begin{aligned} Fe^{ika} &= Ce^{ira} + De^{-ira} = \\ &= Ce^{ira} + C\, e^{2ira}\, \frac{r-k}{r+k}\, e^{-ira} = \\ &= Ce^{ira} \left(1 + \frac{r-k}{r+k}\right) = Ce^{ira} \frac{2r}{r+k} \end{aligned}$$

Appendix4: Calculation of A,B,C,D,F for the finite potential well

1. Let us consider, now, the two equations concerning the continuity conditions of ψ and $\psi^{'}$ in $x = -a$. We have

$$Ae^{-ika} + Be^{ika} = Ce^{-ira} + De^{ira}$$
$$kAe^{-ika} - kBe^{ika} = rCe^{-ira} - rDe^{ira}$$

2. If we multiply the first equation for r, we obtain

$$rAe^{-ika} + rBe^{ika} = rCe^{-ira} + rDe^{ira}$$
$$kAe^{-ika} - kBe^{ika} = rCe^{-ira} - rDe^{ira}$$

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1. and if we add them, we get

$$Ae^{-ika}(r+k) + Be^{ika}(r-k) = 2r\,Ce^{-ira}$$

2. whereas, if we subract them, we obtain

$$Ae^{-ika}(r-k) + Be^{ika}(r+k) = 2r\,De^{ira} =$$

$$= 2rCe^{2ira}\frac{r-k}{r+k}e^{ira} =$$

$$= 2r\,Ce^{-ira}\; e^{4ira}\frac{r-k}{r+k}$$

$$\Rightarrow (r+k)e^{-2ira}\left[Ae^{-ika}(r-k) + Be^{ika}(r+k)\right] =$$

$$= (r-k)e^{2ira}\left[Ae^{-ika}(r+k) + Be^{ika}(r-k)\right]$$

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from which we have

$$e^{-2ira}\left[Ae^{-ika}(r^2-k^2)+Be^{ika}(r+k)^2\right]=$$
$$=e^{2ira}\left[Ae^{-ika}(r^2-k^2)+Be^{ika}(r-k)^2\right]$$
$$\Rightarrow e^{-2ira}\left[Ae^{-ika}(r^2-k^2)+Be^{ika}(r^2+k^2)+2Be^{ika}rk\right]=$$
$$=e^{2ira}\left[Ae^{-ika}(r^2-k^2)+Be^{ika}(r^2+k^2)-2Be^{ika}rk\right]$$
$$\Rightarrow A(r^2-k^2)e^{-ika}2i\,sin(2ra)+$$
$$+B(r^2+k^2)e^{ika}2i\,sin(2ra)-2B\,rk\,e^{ika}\,2\,cos(2ra)=0$$

and finally

$$B=i\,A\,e^{-2ika}\,\frac{(r^2-k^2)sin(2ra)}{2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)}$$

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Concerning the parameter C, we have

$$2r\, C\, e^{-ira} = Ae^{-ika}(r+k) + Be^{ika}(r-k) =$$
$$= Ae^{-ika}(r+k) +$$
$$+i\, A\, e^{-ika}\, \frac{(r^2-k^2)sin(2ra)(r-k)}{2rk\, cos(2ra) - i(r^2+k^2)sin(2ra)}$$

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and therefore

$$2r\,C\,e^{-ira} = Ae^{-ika}\cdot$$
$$= \cdot\left\{\frac{(r+k)\left[2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)\right]}{2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)}+\right.$$
$$\left.+\frac{i(r^2-k^2)sin(2ra)(r-k)}{2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)}\right\} =$$
$$= Ae^{-ika}\frac{(r+k)2kr\,cos(2ra)-2ikr(k+r)sin(2ra)}{2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)} =$$
$$= Ae^{-ika}\frac{(r+k)2kr\,e^{-2ira}}{2rk\,cos(2ra)-i(r^2+k^2)sin(2ra)}$$

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In summary, about C, we have

$$2r\,C\,e^{-ira} = A\,\frac{e^{-ika} \quad (r+k)2kr\,e^{-2ira}}{2rk\,cos(2ra) - i(r^2+k^2)sin(2ra)}$$

and finally

$$C = A\,\frac{e^{-ira}\,e^{-ika} \quad (r+k)k}{2rk\,cos(2ra) - i(r^2+k^2)sin(2ra)}$$

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Using the above result we can rewrite the coefficient F and we have

$$F = C\,e^{ira}\frac{2r}{r+k} =$$
$$= A\,e^{-ika}\frac{2kr}{2rk\,cos(2ra) - i(r^2+k^2)sin(2ra)}$$

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