QUANTUM MECHANICS
Appendix 3

Finite square well: Wave function normalization

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Appendix3: calculation of the normalization constants A and B

1. Let us start from the even solutions that we have obtained for the (bound) stationary states associated to the finite square well. We have found

$$|x| \geq a \quad : \quad \psi(x) = \alpha\, e^{-r|x|}$$
$$x < a \quad : \quad \psi(x) = B\, cos(kx)$$

where $\alpha = B\, e^{\eta}\, cos\xi$ and $\eta = a\, r = \xi\, tg\xi$.

2. We intend to determine the constant B for which $|\psi|^2$ is normalized (B is unique, a part a possible complex phase ...).

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2. We intend to determine the constant B for which $|\psi|^2$ is normalized (B is unique, a part a possible complex phase ...).

Appendix3: calculation of the normalization constants A and B

Since $|\psi|^2$ is even, we will integrate only for $x > 0$ and then double the result. We have

$$\int_0^a dx\,|\psi(x)|^2 = B^2 \int_0^a dx\,cos^2(kx) =$$

$$\begin{aligned}
&= \frac{1}{2}B^2 \int_0^a dx\,[1 + cos(2kx)] = \\
&= B^2\left[\frac{a}{2} + \frac{1}{2}\frac{sin(2ka)}{2k}\right] = \\
&= B^2 \frac{2ak + sin(2ak)}{4k} = B^2\,\frac{a}{2}\,\frac{2\xi + sin(2\xi)}{2\xi} = \\
&= B^2\,\frac{a}{2}\left(1 + \frac{sin\xi\,cos\xi\,{\color{red}tg\xi}}{\xi\,{\color{red}tg\xi}}\right) = \\
&= B^2\,\frac{a}{2}\left(1 + \frac{sin^2\xi}{\eta}\right)
\end{aligned}$$

Appendix3: calculation of the normalization constants A and B

1. From the region in which $x > a$, we get

$$\int_a^\infty dx\, |\psi(x)|^2 = \alpha^2 \int_a^\infty dx\, e^{-2rx} =$$

$$= \alpha^2 \left(-\frac{1}{2r}\right) e^{-2rx}\Big|_a^\infty =$$

$$= \alpha^2 \frac{1}{2r} e^{-2ar} = a\,\alpha^2\, \frac{e^{-2\eta}}{2\eta}$$

2. But

$$\alpha = B\, e^{\eta}\, cos\xi \Rightarrow \int_a^\infty dx\, |\psi(x)|^2 =$$

$$= a\, B^2\, e^{2\eta}\, cos^2\xi\, \frac{e^{-2\eta}}{2\eta} = \frac{a}{2}\, B^2\, \frac{cos^2\xi}{\eta}$$

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In conclusion, for the even solutions we have

$$\int_{-\infty}^{+\infty} dx\, |\psi(x)|^2 = a\, B^2 \left[\left(1 + \frac{sin^2\xi}{\eta} \right) + \frac{cos^2\xi}{\eta} \right] =$$

$$= \; a\, B^2 \left(1 + \frac{1}{\eta} \right)$$

and therefore

$$\int_{-\infty}^{+\infty} dx\, |\psi(x)|^2 = 1 \;\; \Rightarrow$$

$$\Rightarrow \;\; B^{-1} = \sqrt{a \left(1 + \frac{1}{\eta} \right)}$$

Appendix3: calculation of the normalization constants A and B

Let us come, now, to consider the odd solutions, for which

$$
\begin{aligned}
x \geq a \quad &: \quad \psi(x) = \alpha\, e^{-rx} \\
x \leq -a \quad &: \quad \psi(x) = -\alpha\, e^{rx} \\
x < a \quad &: \quad \psi(x) = A\, sin(kx)
\end{aligned}
$$

where $\alpha = A\, e^{\eta}\, sin\xi$ and $\eta = kr = -\xi\, ctg\xi$.

Appendix3: calculation of the normalization constants A and B

Since also in this case $|\psi|^2$ is **even**, we will integrate only for $x > 0$ and then double the result. We have

$$\int_0^a dx\,|\psi(x)|^2 = A^2 \int_0^a dx\,sin^2(kx) =$$

$$= \frac{1}{2}A^2 \int_0^a dx\,[1 - cos(2kx)] =$$

$$= A^2\left[\frac{a}{2} - \frac{1}{2}\frac{sin(2ka)}{2k}\right] = A^2\,\frac{a}{2}\left(1 - \frac{2sin\xi\,cos\xi}{2\xi}\right) =$$

$$= A^2\,\frac{a}{2}\left(1 - \frac{sin\xi\,cos\xi(-ctg\xi)}{\xi(-ctg\xi)}\right) =$$

$$= A^2\,\frac{a}{2}\left(1 + \frac{cos^2\xi}{\eta}\right)$$

Appendix3: calculation of the normalization constants A and B

1. From the $x > a$ region, we get again

$$\int_a^\infty dx\,|\psi(x)|^2 = \alpha^2 \int_a^\infty dx\, e^{-2rx} =$$
$$= \alpha^2 \left(-\frac{1}{2r}\right) e^{-2rx}\Big|_a^\infty =$$
$$= \alpha^2 \frac{1}{2r} e^{-2ar} = a\,\alpha^2\,\frac{e^{-2\eta}}{2\eta}$$

2. But now

$$\alpha = A\,e^{\eta}\,sin\xi \Rightarrow \int_a^\infty dx\,|\psi(x)|^2 =$$
$$= a\,A^2\,e^{2\eta}\,sin^2\xi\,\frac{e^{-2\eta}}{2\eta} = \frac{a}{2}\,A^2\,\frac{sin^2\xi}{\eta}$$

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$$\alpha = A\, e^{\eta}\, sin\xi \;\Rightarrow\; \int_a^\infty dx\,|\psi(x)|^2 =$$
$$= a\, A^2\, e^{2\eta}\, sin^2\xi\, \frac{e^{-2\eta}}{2\eta} = \frac{a}{2}\, A^2\, \frac{sin^2\xi}{\eta}$$

Appendix3: calculation of the normalization constants A and B

In conclusion, also for the odd solutions we have

$$\int_{-\infty}^{+\infty} dx\,|\psi(x)|^2 = a\,A^2\,\left[\left(1+\frac{cos^2\xi}{\eta}\right)+\frac{sin^2\xi}{\eta}\right] =$$

$$= \; a\,A^2\,\left(1+\frac{1}{\eta}\right)$$

and therefore, again

$$\int_{-\infty}^{+\infty} dx\,|\psi(x)|^2 = 1 \;\; \Rightarrow$$

$$\Rightarrow \;\; A^{-1} = \sqrt{a\left(1+\frac{1}{\eta}\right)}$$

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