QUANTUM MECHANICS
Lecture 2

The Schrödinger equation
The statistical interpretation
The probability

Enrico Iacopini

September 4, 2019

D. J. Griffiths: paragraphs 1.1, 1.2 and 1.3

Why QM ?

1. In the previous lecture we have seen the reasons why Classical Phyics had to be rewieved and a new Mechanics was necessary ($\Rightarrow QM$).

2. We have also said that QM **represents a revolutionary departure from classical ideas and in many aspects it appears counterintuitive.**

3. Nevertheless, up to now, the QM **predictions** have always been found **in perfect agreement** with the **experiments !**

Why QM ?

1. In the previous lecture we have seen the reasons why Classical Phyics had to be rewieved and a new Mechanics was necessary ($\Rightarrow QM$).

2. We have also said that QM **represents a revolutionary departure from classical ideas and in many aspects it appears counterintuitive.**

3. Nevertheless, up to now, the QM **predictions** have always been found **in perfect agreement** with the **experiments !**

Why QM ?

1. In the previous lecture we have seen the reasons why Classical Phyics had to be rewieved and a new Mechanics was necessary ($\Rightarrow QM$).
2. We have also said that QM **represents a revolutionary departure from classical ideas and in many aspects it appears counterintuitive.**
3. Nevertheless, up to now, the QM **predictions** have always been found **in perfect agreement** with the **experiments !**

The Schrödinger equation

1. In Quantum Mechanics, the particle dynamics is described by the Schrödinger equation, that, in one dimension, reads

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\,\Psi$$

2. This equation allow to evaluate the time evolution of the **"wave-function"** $\Psi = \Psi(x,t)$, that describes the physical state of the system under consideration.

The Schrödinger equation

1. In Quantum Mechanics, the particle dynamics is described by the Schrödinger equation, that, in one dimension, reads

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\,\Psi$$

2. This equation allow to evaluate the time evolution of the **"wave-function"** $\Psi = \Psi(x, t)$, that describes the physical state of the system under consideration.

The wave function

1. The Schrödinger equation is the QM counterpart of the Newton's $Lex\,II$:
2. given the initial conditions and the force law (the potential energy $V(x)$ ...), in CM, $Lex\,II$ allow us to find the particle position $x = x(t)$ as a function of time;
3. similarly, in QM, the Schrödinger equation allow us to determine the wave function $\Psi(x,t)$ at any time.
4. However, everybody undestands the meaning of the trajectory equation $x = x(t)$; but **which is the meaning of the wave-function** $\Psi(x,t)$ appearing in the Schrödinger equation ?

The wave function

1. The Schrödinger equation is the QM counterpart of the Newton's $Lex\, II$:
2. given the initial conditions and the force law (the potential energy $V(x)$ ...), in CM, $Lex\, II$ allow us to find the particle position $x = x(t)$ as a function of time;
3. similarly, in QM, the Schrödinger equation allow us to determine the wave function $\Psi(x, t)$ at any time.
4. However, everybody undestands the meaning of the trajectory equation $x = x(t)$; but **which is the meaning of the wave-function** $\Psi(x, t)$ appearing in the Schrödinger equation ?

The wave function

1. The Schrödinger equation is the QM counterpart of the Newton's $Lex\,II$:
2. given the initial conditions and the force law (the potential energy $V(x)$ ...), in CM, $Lex\,II$ allow us to find the particle position $x = x(t)$ as a function of time;
3. similarly, in QM, the Schrödinger equation allow us to determine the wave function $\Psi(x, t)$ at any time.
4. However, everybody undestands the meaning of the trajectory equation $x = x(t)$; but **which is the meaning of the wave-function** $\Psi(x, t)$ appearing in the Schrödinger equation ?

The wave function

1. The Schrödinger equation is the QM counterpart of the Newton's $Lex\,II$:
2. given the initial conditions and the force law (the potential energy $V(x)$ ...), in CM, $Lex\,II$ allow us to find the particle position $x = x(t)$ as a function of time;
3. similarly, in QM, the Schrödinger equation allow us to determine the wave function $\Psi(x, t)$ at any time.
4. However, everybody undestands the meaning of the trajectory equation $x = x(t)$; but **which is the meaning of the wave-function** $\Psi(x, t)$ appearing in the Schrödinger equation ?

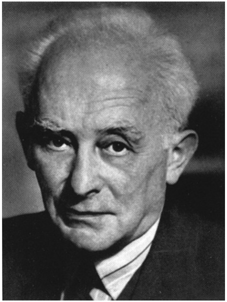
The wave function

It is precisely in the interpretation of the wave-function that there is **one of the most important differences with respect to the classical world.**

According to Max Born

$|\mathbf{\Psi}(\mathbf{x},\mathbf{t})|^2$ is proportional to the **probability density** of finding the particle in the position x at time t.

It is the so called **Copenaghen or statistical interpretation.**



The wave function

1. More precisely, according to this interpretation, the probability of finding the particle between a and b at time t is given by

$$P(a,b;t) = \frac{\int_a^b dx\,|\Psi(x,t)|^2}{\int_{-\infty}^{+\infty} dx\,|\Psi(x,t)|^2}$$

2. This, in particular, requires that $\Psi(x,t)$ is a **square-integrable function**, or, in other words, that

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx = A(t)$$

for some positive real time function $A(t)$.

The wave function

1. More precisely, according to this interpretation, the probability of finding the particle between a and b at time t is given by

$$P(a,b;t) = \frac{\int_a^b dx\, |\Psi(x,t)|^2}{\int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2}$$

2. This, in particular, requires that $\Psi(x,t)$ is a **square-integrable function**, or, in other words, that

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx = A(t)$$

for some positive real time function $A(t)$.

The wave function

1. According to QM, a physical state is **fully described by its wave-function** Ψ; although the correspondence is not one-to-one: an axiom of QM states that wave functions differing only by a complex multiplicative constant (different from zero) **describe** the ***same*** physical state.

   $$\Psi(x,t) \;\; and \;\; K \cdot \Psi(x,t) \; are \; equivalent$$

   for any (non null) complex constant K.

2. This aspect has to do with the so-called ***normalization*** of the wave function, and we will come on this point later on.

The wave function

1. According to QM, a physical state is **fully described by its wave-function** Ψ; although the correspondence is not one-to-one: an axiom of QM states that wave functions differing only by a complex multiplicative constant (different from zero) **describe** the ***same*** physical state.

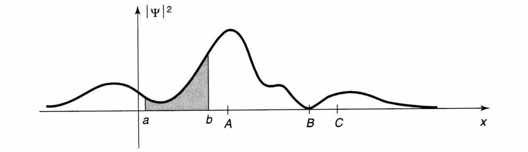
   $$\Psi(x,t) \;\; and \;\; K \cdot \Psi(x,t) \; are \; equivalent$$

   for any (non null) complex constant K.

2. This aspect has to do with the so-called ***normalization*** of the wave function, and we will come on this point later on.

Consequeces from the statistical interpretation

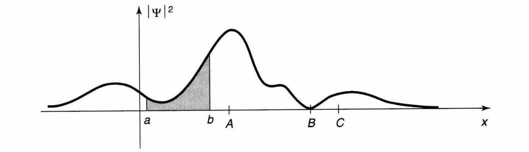
1. The most obvious consequence coming from the statistical interpretation is that the physical state af a particle, (fully) described by wave function Ψ, has an **intrinsic, unavoidable indeterminacy**.



2. The particle will be found very likely near A and never in B; but if we measure the particle position many times, **starting from the same physical state**, we will possibly get, each time, **different results !**

Consequeces from the statistical interpretation

1. The most obvious consequence coming from the statistical interpretation is that the physical state af a particle, (fully) described by wave function Ψ, has an **intrinsic, unavoidable indeterminacy**.



2. The particle will be found very likely near A and never in B; but if we measure the particle position many times, **starting from the same physical state**, we will possibly get, each time, **different results !**

Consequeces

1. Assume now that, after a position measurement, we have got for the particle the position C. **Can we say where** the particle was just **before the measurement ?**

2. QM states that, before the measurement, since the state described by ψ implies only that there is a non-null probability, but not a certainty, to find the particle in C, *we cannot say* ***nothing*** *about the particle position.*

3. **This looks crazy, but it is like that !** Until a measurement is done, the particle is potentially where the wave function indicates a non-null probability.

Consequeces

1. Assume now that, after a position measurement, we have got for the particle the position C. **Can we say where** the particle was just **before the measurement ?**

2. QM states that, before the measurement, since the state described by ψ implies only that there is a non-null probability, but not a certainty, to find the particle in C, *we cannot say* ***nothing*** *about the particle position.*

3. **This looks crazy, but it is like that !** Until a measurement is done, the particle is potentially where the wave function indicates a non-null probability.

Consequeces

1. Assume now that, after a position measurement, we have got for the particle the position C. **Can we say where** the particle was just **before the measurement ?**

2. QM states that, before the measurement, since the state described by ψ implies only that there is a non-null probability, but not a certainty, to find the particle in C, *we cannot say **nothing** about the particle position.*

3. **This looks crazy, but it is like that !**
   Until a measurement is done, the particle is potentially where the wave function indicates a non-null probability.

Consequeces

1. And what about the position where will be found the particle if we perform another position measurement, just after the previous one that has found the particle in C ?

2. Here QM states that, if we repeat the measurement immediately after, we will find again the same result as before, so **still in position** C.

3. ***And this, at least, sounds quite reasonable !***

Consequeces

1. And what about the position where will be found the particle if we perform another position measurement, just after the previous one that has found the particle in C ?
2. Here QM states that, if we repeat the measurement immediately after, we will find again the same result as before, so **still in position** C.
3. *And this, at least, sounds quite reasonable* !

Consequeces

1. And what about the position where will be found the particle if we perform another position measurement, just after the previous one that has found the particle in C ?

2. Here QM states that, if we repeat the measurement immediately after, we will find again the same result as before, so **still in position** C.

3. ***And this, at least, sounds quite reasonable !***

Consequeces

1. However this "*reasonable conclusion*" means that **the measurement has changed the physical state** of the particle and, as a consequence, also its wave function

$$\mathbf{\Psi} \rightarrow \mathbf{\Psi}'$$

2. In fact, now, the new wave function Ψ' must be such to give a probability peaked in C, with a virtually absence of spread around this position.

3. It is the so called ***collapse of the wave function***, another prediction of QM which was (and still is) not easy to digest.

Consequeces

1. However this "*reasonable conclusion*" means that **the measurement has changed the physical state** of the particle and, as a consequence, also its wave function

   $$\boldsymbol{\Psi} \rightarrow \boldsymbol{\Psi}'$$

2. In fact, now, the new wave function Ψ' must be such to give a probability peaked in C, with a virtually absence of spread around this position.

3. It is the so called ***collapse of the wave function***, another prediction of QM which was (and still is) not easy to digest.

Consequeces

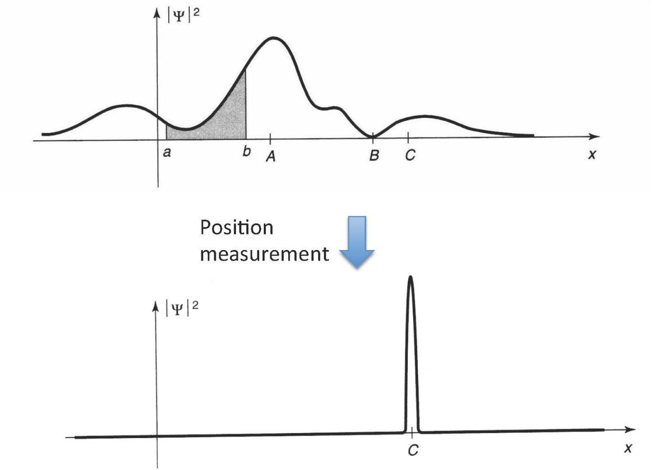
1. However this "*reasonable conclusion*" means that **the measurement has changed the physical state** of the particle and, as a consequence, also its wave function

   $$\boldsymbol{\Psi} \rightarrow \boldsymbol{\Psi}'$$

2. In fact, now, the new wave function Ψ' must be such to give a probability peaked in C, with a virtually absence of spread around this position.

3. It is the so called ***collapse of the wave function***, another prediction of QM which was (and still is) not easy to digest.

The w.f. collapse



The probability

Because of the statistical interpretation of $|\Psi|^2$,
it may be useful some reminder concerning the Probability Theory ...

Probability

Assume you have a perfect dice.

Which is the probability of getting, f.i., a 5 after a roll ?

1. Everyone knows that this probability is $P = 1/6$.
2. The reason of this conclusion is that there are **six** different possibilities that should appear with the same frequency, and **only one** corresponds to the 5.

Probability

Assume you have a perfect dice.

Which is the probability of getting, f.i., a 5 after a roll ?

1. Everyone knows that this probability is $\mathbf{P = 1/6}$.
2. The reason of this conclusion is that there are **six** different possibilities that should appear with the same frequency, and **only one** corresponds to the 5.

Probability

Assume you have a perfect dice.

Which is the probability of getting, f.i., a 5 after a roll ?

1. Everyone knows that this probability is $\mathbf{P = 1/6}$.
2. The reason of this conclusion is that there are **six** different possibilities that should appear with the same frequency, and **only one** corresponds to the 5.

Probability

1. The probability of the event is identified with the **ratio** between the number of "*positive*" cases and their total number.
2. Clearly, if instead of 5, we bet on any another number between 1 and 6, the probability remains 1/6 ...

Probability

1. The probability of the event is identified with the **ratio** between the number of "*positive*" cases and their total number.

2. Clearly, if instead of 5, we bet on any another number between 1 and 6, the probability remains 1/6 ...

Probability

1. But what about the probability that, with **two** dice, we obtain 12 as sum of the two ?

2. Since the possibilities are now $6 \times 6 = 36$, and only in one case we can get 12 (with both dice giving a 6), the probability is 1/36.

Probability

1. But what about the probability that, with **two** dice, we obtain 12 as sum of the two ?

2. Since the possibilities are now $6 \times 6 = 36$, and only in one case we can get 12 (with both dice giving a 6), the probability is $1/36$.

Probability

And in case we bet on 11, the probability still remains 1/36 ?

1. No.
2. Because now there are two possibilities out of 36 to get 11, $(6, 5)$ and $(5, 6)$, therefore the probability becomes $2/36 = 1/18$.

Probability

And in case we bet on 11, the probability still remains 1/36 ?

1. **No**.
2. Because now there are two possibilities out of 36 to get 11, $(6, 5)$ and $(5, 6)$, therefore the probability becomes $2/36 = 1/18$.

Probability

And in case we bet on 11, the probability still remains 1/36 ?

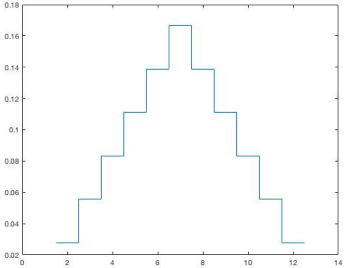
1. **No**.
2. Because now there are two possibilities out of 36 to get 11, $(6, 5)$ and $(5, 6)$, therefore the probability becomes $2/36 = 1/18$.

Probability

Probability of obtaining any sum from **2** to **12**:

$$\begin{aligned}
\mathbf{12}&: (6,6) & \mathbf{P} = \mathbf{1/36}\\
\mathbf{11}&: (6,5),(5,6) & \mathbf{P} = \mathbf{2/36}\\
\mathbf{10}&: (6,4),(4,6),(5,5) & \mathbf{P} = \mathbf{3/36}\\
\mathbf{9}&: (6,3),(3,6),(5,4),(4,5) & \mathbf{P} = \mathbf{4/36}\\
\mathbf{8}&: (6,2),(2,6),(5,3),(3,5),(4,4) & \mathbf{P} = \mathbf{5/36}\\
\mathbf{7}&: (6,1),(1,6),(5,2),(2,5),(4,3),(3,4) & \mathbf{P} = \mathbf{6/36}\\
\mathbf{6}&: (5,1),(1,5),(4,2),(2,4),(3,3) & \mathbf{P} = \mathbf{5/36}\\
\mathbf{5}&: (4,1),(1,4),(3,2),(2,3) & \mathbf{P} = \mathbf{4/36}\\
\mathbf{4}&: (3,1),(1,3),(2,2) & \mathbf{P} = \mathbf{3/36}\\
\mathbf{3}&: (2,1),(1,2) & \mathbf{P} = \mathbf{2/36}\\
\mathbf{2}&: (1,1) & \mathbf{P} = \mathbf{1/36}
\end{aligned}$$

Probability distribution



The probability distribution shows that the most probable value if **7**, with a probability equal to $\mathbf{P = 1/6}$.

Probability distribution

1. Associated to the probability distribution, we can define the **average (mean value)** m as

$$m \equiv \sum_{s=2}^{12} s \cdot P(s) \equiv < s >$$

2. It is easy to see that, in our case, we obtain $m = 7$.
3. But be careful: ***the average and the most probable value are a priori independent !***

Probability distribution

1. Associated to the probability distribution, we can define the **average (mean value)** m as

$$m \equiv \sum_{s=2}^{12} s \cdot P(s) \equiv < s >$$

2. It is easy to see that, in our case, we obtain $m = 7$.
3. But be careful: ***the average and the most probable value are a priori independent !***

Probability distribution

1. Associated to the probability distribution, we can define the **average (mean value)** m as

$$m \equiv \sum_{s=2}^{12} s \cdot P(s) \equiv < s >$$

2. It is easy to see that, in our case, we obtain $m = 7$.
3. But be careful: ***the average and the most probable value are a priori independent !***

Probability distribution

Take, f.i., the following probability distribution:

$$\begin{aligned} P(1) &= 1/2 \\ P(2) &= 1/4 \\ P(3) &= 1/8 \\ P(4) &= 1/8 \end{aligned}$$

the most probable value is clearly 1,
but the average value is

$$\begin{aligned} m &= \sum_{s=1,4} P(s) \cdot s = \\ &= \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 4 = \frac{15}{8} \end{aligned}$$

Probability distribution

1. Together with the average $m \equiv < s >$, another important quantity, which gives a measure of the **spread** of the probability distribution, is the ***variance*** D defined as

$$D \equiv \sum_s P(s)\Big(s - < m >\Big)^2$$

2. The square root of the variance is called the ***standard deviation*** σ ***or the root mean square (r.m.s.)*** of the distribution

$$\sigma \equiv \sqrt{D} = \sqrt{\sum_s P(s)\Big(s - < m >\Big)^2}$$

Probability distribution

1. Together with the average $m \equiv < s >$, another important quantity, which gives a measure of the spread of the probability distribution, is the ***variance D*** defined as

$$D \equiv \sum_s P(s)\Big(s - < m >\Big)^2$$

2. The square root of the variance is called the ***standard deviation σ or the root mean square (r.m.s.)*** of the distribution

$$\sigma \equiv \sqrt{D} = \sqrt{\sum_s P(s)\Big(s - < m >\Big)^2}$$

Probability distribution

We have (remember that $\sum_s P(s) = 1$ ...)

$$\sigma^2 = \sum_s P(s)\left(s^2 - 2s < m > + < m >^2\right) =$$

$$= \left(\sum_s P(s)\, s^2\right) - 2 < m >< m > + < m >^2 =$$

$$= < s^2 > - < m >^2$$

Probability distribution

In case of a **continuous random variable**, the probability distribution is described by a probability density function (pdf) $\underline{\rho(x)}$ such that

1. $\rho(x)\,dx$ is the probability to find the random variable between x and $x + dx$;
2. x varies between some x_{min} and x_{max} (that can also be equal to $\pm\infty$, respectively ...)
3. $\rho(x) \geq 0$

$$\int_{x_{min}}^{x_{max}} \rho(x)\,dx = 1$$

Probability distribution

In case of a **continuous random variable**, the probability distribution is described by a probability density function (pdf) $\underline{\rho(x)}$ such that

1. $\rho(x)\,dx$ is the probability to find the random variable between x and $x + dx$;
2. x varies between some x_{min} and x_{max} (that can also be equal to $\pm\infty$, respectively ...)
3. $\rho(x) \geq 0$

$$\int_{x_{min}}^{x_{max}} \rho(x)\,dx = 1$$

Probability distribution

In case of a **continuous random variable**, the probability distribution is described by a probability density function (pdf) $\underline{\rho(x)}$ such that

1. $\rho(x)\,dx$ is the probability to find the random variable between x and $x + dx$;
2. x varies between some x_{min} and x_{max} (that can also be equal to $\pm\infty$, respectively ...)
3. $\rho(x) \geq 0$

$$\int_{x_{min}}^{x_{max}} \rho(x)\,dx = 1$$

Probability distribution

In case of a **continuous random variable**, the probability distribution is described by a probability density function (pdf) $\underline{\rho(x)}$ such that

1. $\rho(x)\,dx$ is the probability to find the random variable between x and $x + dx$;
2. x varies between some x_{min} and x_{max} (that can also be equal to $\pm\infty$, respectively ...)
3. $\rho(x) \geq 0$

$$\int_{x_{min}}^{x_{max}} \rho(x)\,dx = 1$$

Probability distribution

1. In the case of a continuous probability distribution, the average (mean) is defined as

$$m \equiv < x > = \int_{x_{min}}^{x_{max}} dx \; \rho(x) \cdot x$$

2. whereas, for the standard deviation, we have

$$\sigma^2 = \int_{x_{min}}^{x_{max}} dx \left(x - m \right)^2 \rho(x) = $$
$$= \left(\int_{x_{min}}^{x_{max}} dx \, \rho(x) \cdot x^2 \right) - m^2$$

Probability distribution

1. In the case of a continuous probability distribution, the average (mean) is defined as

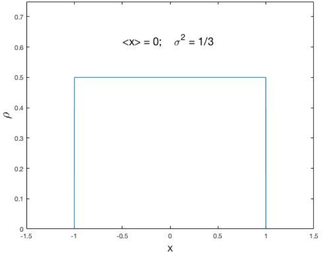
$$m \equiv < x > = \int_{x_{min}}^{x_{max}} dx \; \rho(x) \cdot x$$

2. whereas, for the standard deviation, we have

$$\begin{aligned} \sigma^2 &= \int_{x_{min}}^{x_{max}} dx \left(x - m \right)^2 \rho(x) = \\ &= \left(\int_{x_{min}}^{x_{max}} dx \, \rho(x) \cdot x^2 \right) - m^2 \end{aligned}$$

First example: flat distribution

Assume that $\rho(x)$ is equal to zero for $|x| \geq a$ and that $\rho(x) = \frac{1}{2a}$ when $|x| \leq a$



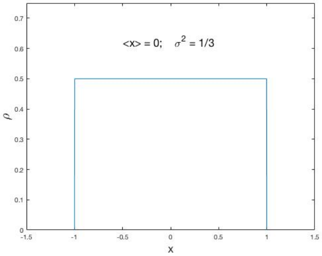
1. Clearly $m \equiv < x > = 0$, but what about σ^2 ?
2. We have

$$\sigma^2 = \int_{-a}^{+a} dx\, \rho(x)\, x^2 = \int_{-a}^{+a} dx\, \frac{1}{2a}\, x^2 =$$

$$= \frac{1}{2a}\left(2\,\frac{a^3}{3}\right) = \frac{a^2}{3} \Rightarrow \sigma = \frac{a}{\sqrt{3}}$$

First example: flat distribution

Assume that $\rho(x)$ is equal to zero for $|x| \geq a$ and that $\rho(x) = \frac{1}{2a}$ when $|x| \leq a$



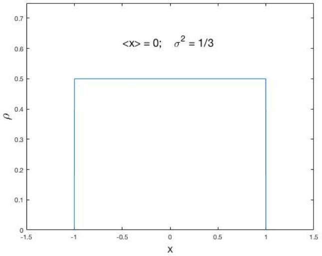
1. Clearly $m \equiv < x > = 0$, but what about σ^2 ?
2. We have

$$\sigma^2 = \int_{-a}^{+a} dx\, \rho(x)\, x^2 = \int_{-a}^{+a} dx\, \frac{1}{2a}\, x^2 =$$

$$= \frac{1}{2a}\left(2\,\frac{a^3}{3}\right) = \frac{a^2}{3} \quad \Rightarrow \quad \sigma = \frac{a}{\sqrt{3}}$$

First example: flat distribution

Assume that $\rho(x)$ is equal to zero for $|x| \geq a$ and that $\rho(x) = \frac{1}{2a}$ when $|x| \leq a$



1. Clearly $m \equiv < x > = 0$, but what about σ^2 ?
2. We have

$$\sigma^2 = \int_{-a}^{+a} dx\, \rho(x)\, x^2 = \int_{-a}^{+a} dx\, \frac{1}{2a}\, x^2 =$$

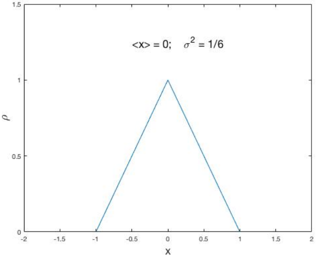
$$= \frac{1}{2a}\left(2\,\frac{a^3}{3}\right) = \frac{a^2}{3} \quad \Rightarrow \quad \sigma = \frac{a}{\sqrt{3}}$$

Triangular prob. density function

Assume now a triangular shape for the p.d.f., such as

$$\rho(x) = \frac{1}{a^2}\Big(a - |x|\Big) \quad for\ |x| \le a$$

$$\rho(x) = 0 \quad otherwise$$



1. The average is still $m = \langle x \rangle = 0$ but now

$$\sigma^2 = \int_{-a}^{a} \rho(x)\cdot x^2\, dx = 2\int_0^a \rho(x)\cdot x^2\, dx =$$

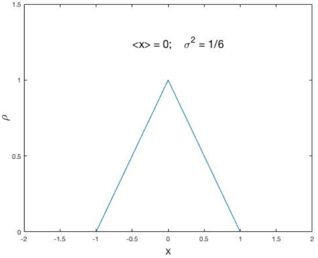
$$= \frac{2}{a^2}\int_0^a \Big(a - x\Big)x^2\, dx = \frac{2}{a^2}\Big(a\frac{a^3}{3} - \frac{a^4}{4}\Big) = \frac{a^2}{6}$$

Triangular prob. density function

Assume now a triangular shape for the p.d.f., such as

$$\rho(x) = \frac{1}{a^2}\Big(a - |x|\Big) \quad \textit{for } |x| \le a$$

$$\rho(x) = 0 \quad \textit{otherwise}$$



1. The average is still $m = < x > = 0$ but now

$$\sigma^2 = \int_{-a}^{a} \rho(x) \cdot x^2\, dx = 2\int_0^a \rho(x) \cdot x^2\, dx =$$
$$= \frac{2}{a^2}\int_0^a \Big(a - x\Big) x^2\, dx = \frac{2}{a^2}\left(a\frac{a^3}{3} - \frac{a^4}{4}\right) = \frac{a^2}{6}$$

So much for the **Probability Theory.**

Let us come back to our **wave functions** ...

The wave function normalization

1. We have said that the solutions of the Schrödinger equation $\Psi(x,t)$ have to be **square integrable**, which means that

$$\forall t : \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = A(t) > 0$$

   where $A(t)$, a priori, can be time dependent (we will see that A is indeed a constant ...).

2. We have also said that, according to QM, two wave functions $\mathbf{\Psi_a}$ and $\mathbf{\Psi_b}$ that are proportional ($\mathbf{\Psi_a = k\,\Psi_b}$), they represent the **same physical state**.

The wave function normalization

1. We have said that the solutions of the Schrödinger equation $\Psi(x,t)$ have to be **square integrable**, which means that

$$\forall t : \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = A(t) > 0$$

where $A(t)$, a priori, can be time dependent (we will see that A is indeed a constant ...).

2. We have also said that, according to QM, two wave functions $\mathbf{\Psi_a}$ and $\mathbf{\Psi_b}$ that are proportional ($\mathbf{\Psi_a = k\, \Psi_b}$), they represent the **same physical state**.

The wave function normalization

1. This allow us to represent any physical state by choosing the(*) particular wave-function which is "normalized" (at least at some time $t = \hat{t}$):

$$\tilde{\Psi}(x,t) = \frac{1}{\sqrt{A(\hat{t})}}\,\Psi(x,t)$$

such that

$$\int_{-\infty}^{+\infty} |\tilde{\Psi}(x,\hat{t})|^2\,dx = 1$$

2. (*) The normalized wave function **is indeed not unique**, but it is determined up to a constant complex phase factor $e^{i\alpha}$.

The wave function normalization

1. This allow us to represent any physical state by choosing the(*) particular wave-function which is "normalized" (at least at some time $t = \hat{t}$):

$$\tilde{\Psi}(x,t) = \frac{1}{\sqrt{A(\hat{t})}}\,\Psi(x,t)$$

such that

$$\int_{-\infty}^{+\infty} |\tilde{\Psi}(x,\hat{t})|^2\,dx = 1$$

2. (*) The normalized wave function **is indeed not unique**, but it is determined up to a constant complex phase factor $e^{i\alpha}$.

Exercise N.1

Assume a wave function

$$\Psi(x,t) = A\, e^{-\mu\,|x|}\; e^{i\,k\,t}$$

where A, λ and ω are positive real constants.

i) Normalize ψ;
ii) Evaluate $< x >$, $< x^2 >$ and σ;
iii) Calculate the probability to find the particle outside the interval $< x > \pm 2\sigma$.